Asymptotic spreading in heterogeneous media

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Joint work with H. Berestycki

Asymptotic spreading in homogeneous media

$$\partial_t u - \Delta u = \mu u(1-u) \tag{1}$$

Theorem

(Aronson-Weinberger 78) Assume that $u_0 = u(0,\cdot)$ is compactly supported and that $0 \leq u_0 \leq 1$, then for all $|e| = 1$:

$$\begin{cases} \lim_{t\to+\infty} u(t, wte) = 1 & \text{if } 0 \leq w < w^*, \\ \lim_{t\to+\infty} u(t, wte) = 0 & \text{if } w > w^*. \end{cases} \tag{2}$$

with $w^* = 2\sqrt{\mu}$. w^* is called the spreading speed *associated with the equation.*

Remark: Possible to get a uniform convergence on $|x| \leq wt$.

Question : generalization to heterogeneous media.

The heterogeneous KPP reaction-diffusion equation

$$\partial_t u - \Delta u = \mu(x)u(1-u) \tag{3}$$

$$0 < \inf_{\mathbb{R}^N} \mu \leq \sup_{\mathbb{R}^N} \mu < +\infty$$

μ uniformly Holder continuous

Remark: Possible to consider heterogeneous diffusion matrix and advection term, more general reaction term, heterogeneity in time.

Definition of the asymptotic spreading speeds

$$\begin{cases} \partial_t u - \Delta u = \mu(x)u(1-u) \text{ in } (0,\infty)\times\mathbb{R}^N, \\ u(0,x) = u_0(x) \text{ compactly supported.} \end{cases} \tag{4}$$

$$w_e^* = \sup\{w \geq 0,\ \forall w' \in [0,w], u(t,w'te) \to 1 \text{ as } t \to +\infty\}$$

$$w_e^{**} = \inf\{w \geq 0,\ \forall w' \geq w, u(t,w'te) \to 0 \text{ as } t \to +\infty\}$$

Remarks:

- $w_e^* = w_e^{**} = 2\sqrt{\mu}$ if μ is homogeneous
- $0 < w_e^* \leq w_e^{**} < +\infty$ (Berestycki-Hamel-N. 08)
- $w_e^* < w_e^{**}$ in general (Berestycki-N 09)

Aim of the talk

$$w_e^* = \sup\{w \geq 0,\ \forall w' \in [0, w], u(t, w'te) \to 1 \text{ as } t \to +\infty\}$$
$$w_e^{**} = \inf\{w \geq 0,\ \forall w' \geq w, u(t, w'te) \to 0 \text{ as } t \to +\infty\}$$

Aim of the talk

Find $\underline{w}_e$ as large as possible and $\overline{w}_e$ as small as possible so that

$$\underline{w}_e \leq w_e^* \leq w_e^{**} \leq \overline{w}_e.$$

Earlier works

Aim: Computation of w_e^* and w_e^{**} in simple heterogeneous unbounded media.

- μ constant at infinity
- μ periodic

Earlier works (1) : μ constant at infinity

$$\partial_t u - \Delta u = \mu(x)u(1-u)$$

$\mu(x) = \mu_0 - b(x)$, $\mu_0 > 0$, $b \geq 0$, b compactly supported.

Theorem

(Berestycki-Hamel-N 08, Berestycki-N 09)

$$w_e^* = w_e^{**} = 2\sqrt{\mu_0}.$$

Interpretation: The spreading speeds only depend on "what happens at infinity".

Earlier works (2): μ periodic

Theorem

Assume that μ is periodic in $x \in \mathbb{R}^N$. For all $e \in \mathbb{S}^{N-1}$, there exists a speed w_e^* such that if u_0 initial datum with compact support,

$$u(t, wte) \to \begin{cases} 1 \text{ if } 0 \leq w < w_e^*, \\ 0 \text{ if } w > w_e^*. \end{cases}$$

In other words, $w_e^* = w_e^{**}$.

Several proofs:

- Gartner-Freidlin 79, Nolen-Xin 08 (probabilistic tools)
- Weinberger 02 (discrete formalism)
- Berestycki-Hamel-N. 08 (periodic eigenvalues+PDE tools)
- Majda-Souganidis 94 (homogenization techniques)

Earlier works (2): space-time periodic coefficients

$$\mathcal{L}\phi = \Delta\phi + \mu(x)\phi$$

$$L_p\phi = e^{-p\cdot x}\mathcal{L}(e^{p\cdot x}\phi) = \Delta\phi + 2p\cdot\nabla\phi + (|p|^2 + \mu(x))\phi$$

Set $\lambda_{per}(L_p)$ the periodic principal eigenvalue of L_p (Krein-Rutman theory):

$$\begin{cases} L_p\phi = \lambda_{per}(L_p)\phi, \\ \phi > 0, \\ \phi \text{ is periodic.} \end{cases} \tag{5}$$

Proposition

$$w_e^* = w_e^{**} = \min_{p\cdot e>0} \frac{\lambda_{per}(L_p)}{p\cdot e}$$

Aim of the talk

$$w_e^* = \sup\{w \geq 0,\ \forall w' \in [0, w], u(t, w'te) \to 1 \text{ as } t \to +\infty\}$$
$$w_e^{**} = \sup\{w \geq 0,\ \forall w' \geq w, u(t, w'te) \to 0 \text{ as } t \to +\infty\}$$

Aim of the talk

Find $\underline{w}_e$ as large as possible and $\overline{w}_e$ as small as possible so that

$$\underline{w}_e \leq w_e^* \leq w_e^{**} \leq \overline{w}_e.$$

One need to take into account:

- only the values of the coefficients at infinity *(cf coefficients that are constant at infinity)*.
- the heterogeneity of these coefficients through "eigenvalues" *(cf periodic coefficients)*.

$$\partial_t u - \Delta u = \mu(x) u(1-u) \tag{6}$$

The main tool: generalized principal eigenvalues

$$\mathcal{L}\phi = \Delta\phi + \mu(x)\phi$$
$$L_p\phi = e^{-p\cdot x}\mathcal{L}(e^{p\cdot x}\phi) = \Delta\phi + 2p\cdot\nabla\phi + (|p|^2 + \mu(x))\phi$$

Definition

The **generalized principal eigenvalues** *associated with the operator $\mathcal{L}$ in the open set $\Omega \subset \mathbb{R}^N$ are:*

$$\begin{array}{lll} \underline{\lambda_1}(L_p,\Omega) & := & \inf\{\lambda \mid \exists\phi \in \mathcal{C}^2(\Omega)\cap W^{1,\infty}(\Omega), \\ & & \inf_\Omega \phi > 0 \textit{ and } L_p\phi \leq \lambda\phi \textit{ in } \Omega\} \\ \overline{\lambda_1}(L_p,\Omega) & := & \sup\{\lambda \mid \exists\phi \in \mathcal{C}^2(\Omega)\cap W^{1,\infty}(\Omega), \\ & & \inf_\Omega \phi > 0 \textit{ and } L_p\phi \geq \lambda\phi \textit{ in } \Omega\}, \end{array} \quad (7)$$

Remark: Similar notions introduced by Berestycki-Nirenberg-Varadhan (94), Berestycki-Hamel-Rossi (07), Berestycki-Rossi (06).

Basic properties of the generalized principal eigenvalues

$$\underline{\lambda_1}(L_p, \Omega) := \inf\{\lambda \mid \exists \phi \in \mathcal{C}^2(\Omega) \cap W^{1,\infty}(\Omega), \inf_\Omega \phi > 0 \text{ and } L_p\phi \leq \lambda\phi \text{ in } \Omega\}$$

$$\overline{\lambda_1}(L_p, \Omega) := \sup\{\lambda \mid \exists \phi \in \mathcal{C}^2(\Omega) \cap W^{1,\infty}(\Omega), \inf_\Omega \phi > 0 \text{ and } L_p\phi \geq \lambda\phi \text{ in } \Omega\}$$

$$\overline{\lambda_1}(L_p, \mathbb{R}^N) \geq \overline{\lambda_1}(L_p, \Omega) \text{ and } \underline{\lambda_1}(L_p, \mathbb{R}^N) \leq \underline{\lambda_1}(L_p, \Omega)$$

Lemma

For all $e \in \mathbb{S}^{N-1}$, if Ω contains balls of arbitrary radius, one has

$$\overline{\lambda_1}(L_p, \Omega) \geq \underline{\lambda_1}(L_p, \Omega)$$

If μ is constant, then $\overline{\lambda_1}(L_p, \mathbb{R}^N) = \underline{\lambda_1}(L_p, \mathbb{R}^N) = \mu + |p|^2$.

If μ is periodic, then, $\overline{\lambda_1}(L_p, \mathbb{R}^N) = \underline{\lambda_1}(L_p, \mathbb{R}^N) = \lambda_{per}(L_p)$.

Analogy with the periodic case

$$\mathcal{L}\phi := \Delta\phi + \mu(x)\phi \text{ and } L_p\phi := e^{-p\cdot x}\mathcal{L}(e^{p\cdot x}\phi)$$

If μ is periodic then $\overline{\lambda_1}(L_p, \mathbb{R}^N) = \underline{\lambda_1}(L_p, \mathbb{R}^N) = \lambda_{per}(L_p)$.
Known result: $w_e^* = w_e^{**} = \min_{p\cdot e>0} \frac{\lambda_{per}(L_p)}{p\cdot e}$.

Question: for general heterogeneous μ, $w_e^* = \min_{p\cdot e>0} \frac{\underline{\lambda_1}(L_p, \mathbb{R}^N)}{p\cdot e}$?

Not optimal! We only need to consider "whats happens at infinity" (cf coefficients that are constant at infinity), i.e. for

$$x > R, R \text{ large in dimension } 1.$$

The main result in dimension 1

$$\overline{H}(p) := \lim_{R\to+\infty} \overline{\lambda_1}(L_p, (R,\infty)) \text{ and } \overline{w} = \min_{p>0} \frac{\overline{H}(p)}{p},$$

$$\underline{H}(p) := \lim_{R\to+\infty} \underline{\lambda_1}(L_p, (R,\infty)) \text{ and } \underline{w} = \min_{p>0} \frac{\underline{H}(p)}{p}.$$

Theorem

(Berestycki-N. 09) Assume $N = 1$. Take u_0 a measurable and compactly supported function such that $0 \leq u_0 \leq 1$ and $u_0 \not\equiv 0$. Then:
1) if $0 \leq w < \underline{w}$, one has $u(t, wt) \to 1$,
2) if $w > \overline{w}$, one has $u(t, wt) \to 0$.

In other words

$$\underline{w} \leq w^* \leq w^{**} \leq \overline{w}.$$

Application: periodic media

We know that $w^* = w^{**} = \min_{p>0} \frac{\lambda_{per}(L_p)}{p}$.

Proposition

$\underline{w} = \overline{w} = \min_{p>0} \frac{\lambda_{per}(L_p)}{p}$

Proof.

- $\overline{\lambda_1}(L_p, \mathbb{R}^N) = \underline{\lambda_1}(L_p, \mathbb{R}^N) = \lambda_{per}(L_p)$
- $\overline{H}(p) = \underline{H}(p) = \lambda_{per}(L_p)$

□

Question: Possible to get $\underline{w} = \overline{w} = w^* = w^{**}$ in other frameworks?

Application: almost periodic media

μ is almost periodic.

Theorem

$$\overline{\lambda_1}(L_p, \mathbb{R}) = \underline{\lambda_1}(L_p, \mathbb{R})$$

Corollary

$$\underline{w} = \overline{w} = \min_{p>0} \frac{\overline{\lambda_1}(L_p, \mathbb{R})}{p}$$

Asymptotically almost periodic media

Assume that there exists μ^* almost periodic such that $\mu(x) \to \mu^*(x)$ unif. in $x > R$ as $R \to +\infty$. Set

$$\mathcal{L}^* = \Delta + \mu^*(x) \text{ and } L_p^* \phi = e^{-p \cdot x} \mathcal{L}^* (e^{p \cdot x} \phi).$$

Proposition

$$\underline{w} = \overline{w} = \min_{p>0} \frac{\overline{\lambda_1}(L_p^*, \mathbb{R})}{p}$$

Proof.

$$\overline{H}(p) = \lim_{R \to +\infty} \overline{\lambda_1}(L_p, (R, \infty)) = \overline{\lambda_1}(L_p^*, \mathbb{R})$$

$$\underline{H}(p) = \lim_{R \to +\infty} \underline{\lambda_1}(L_p, (R, \infty)) = \underline{\lambda_1}(L_p^*, \mathbb{R})$$

□

Generalization to higher dimension: a first result

For all $e \in \mathbb{S}^{N-1}$ and $p \in \mathbb{R}^N$,

$$\overline{G}(e,p) := \lim_{R\to+\infty} \overline{\lambda_1}(L_p, \{|x| > R\}),$$

$$\underline{G}(e,p) := \lim_{R\to+\infty} \underline{\lambda_1}(L_p, \{|x| > R\}).$$

If μ is periodic $w_e^* = w_e^{**} = \min_{p\cdot e>0} \frac{\lambda_{per}(L_p)}{p\cdot e}$. Define by analogy

$$\underline{v}_e := \min_{p\cdot e>0} \frac{\underline{G}(e,p)}{p\cdot e}, \overline{v}_e := \min_{p\cdot e>0} \frac{\overline{G}(e,p)}{p\cdot e}$$

Generalization to higher dimension: a first result

$$\overline{G}(p) := \lim_{R\to+\infty} \overline{\lambda_1}(L_p, \{|x| > R\}),$$

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$$\underline{v}_e := \min_{p\cdot e>0} \frac{\underline{G}(p)}{p\cdot e}, \overline{v}_e := \min_{p\cdot e>0} \frac{\overline{G}(p)}{p\cdot e}$$

Proposition

One has

$$\underline{v}_e \leq w_e^* \leq w_e^{**} \leq \overline{v}_e.$$

Not optimal!

Counter-example: $\partial_t u - \Delta u = \mu(x)u(1-u)$ in $\mathbb{R}^2$, $\mu(x) = \mu_+$ if $x_1 > 0$, μ_- if $x_1 \leq 0$, $\mu_+ > \mu_-$.
$\underline{v}_e = 2\sqrt{\mu_-}$ and $\overline{v}_e = 2\sqrt{\mu_+}$.

Meaning of “what happens at infinity”

Question: What is the meaning of “what happens at infinity” in dimension N?

$$C_{R,\alpha}(e) = \{x \in \mathbb{R}^N, |x| > R, |\frac{x}{|x|} - e| < \alpha\}$$

$$\overline{H}(e,p) := \lim_{\alpha\to 0, R\to +\infty} \overline{\lambda_1}(L_p, C_{R,\alpha}(e)),$$

$$\underline{H}(e,p) := \lim_{\alpha\to 0 R\to +\infty} \underline{\lambda_1}(L_p, C_{R,\alpha}(e)).$$

$$\underline{w}_e := \min_{p\cdot e>0} \frac{\underline{H}(e,p)}{p\cdot e}, \overline{w}_e := \min_{p\cdot e>0} \frac{\overline{H}(e,p)}{p\cdot e}$$

Question: $\underline{v}_e \leq w_e^* \leq w_e^{**} \leq \overline{v}_e$? No!

Properties of the Hamiltonians

Proposition

$(e,p) \in \mathbb{S}^{N-1} \times \mathbb{R}^N \mapsto \overline{H}(e,p)$ and $(e,p) \mapsto \underline{H}(e,p)$ are continuous in p and $\exists c, C > 0$ independent of e such that

$$c(1+|p|^2) \leq \underline{H}(e,p) \leq \overline{H}(e,p) \leq C(1+|p|^2).$$

Not continuous in e in general, only locally bounded $\Rightarrow$ No comparison result, no uniqueness...

The main result in dimension N

Proposition

There exist $\overline{Z}$ upper semi-continuous and $\underline{Z}$ lower semi-continuous that are maximal (resp. minimal) discontinuous viscosity solutions *of*

$$\begin{cases} \max\{\partial_t \overline{Z} - \overline{H}(\frac{x}{|x|}, \nabla \overline{Z}), \overline{Z}\} = 0 \text{ in } (0,\infty) \times (\mathbb{R}^N \backslash \{0\}), \\ \max\{\partial_t \underline{Z} - \underline{H}(\frac{x}{|x|}, \nabla \underline{Z}), \underline{Z}\} = 0 \text{ in } (0,\infty) \times (\mathbb{R}^N \backslash \{0\}), \\ \lim_{t\to 0} \overline{Z}(t,x) = \lim_{t\to 0} \underline{Z}(t,x) = -\infty \text{ if } x \neq 0, 0 \text{ if } x = 0, \\ \overline{Z}(t,0) = \underline{Z}(t,0) = 0 \text{ for all } t > 0. \end{cases} \quad (8)$$

Proof. Coercivity of the Hamiltonian + Perron's method (Ishii 87)+ comparison results on smoothened HJ equations. □

The main result in dimension N

Theorem

(Berestycki-N 09) Take u_0 a measurable and compactly supported function such that $0 \leq u_0 \leq 1$ and $u_0 \not\equiv 0$. Then:
1) if $we \in Int\{\underline{Z} = 0\}$, one has $u(t, wte) \to 1$,
2) if $\overline{Z}(1, we) < 0$, one has $u(t, wte) \to 0$.

Applications: Almost periodic coefficients, asymptotically almost periodic coefficients. $\mu =$Heavyside .