1. Introduction. Consider an arbitrage-free market described by a probability measure $\mathbb{P}$ on a scenario space $(\Omega, \mathcal{A})$. There is a risk-free asset whose price at time $\tau$ is $e^{r\tau}$, $r \geq 0$ and a risky asset whose price at time $\tau$ is $S_\tau$. Specifying an arbitrage-free option pricing model necessitates the choice of a risk-neutral measure, i.e. a probability $\mathbb{P}^*$ equivalent to $\mathbb{P}$ such that the discounted price $(e^{-r\tau}S_\tau)_{\tau \in [0,T]}$ is a martingale under $\mathbb{P}^*$. Such a probability measure $\mathbb{P}^*$ allows for the pricing of European options; consider a European option with payoff $\overline{P}_\tau$ at maturity $t \leq T$: its price at time $\tau \leq t$ is $P_\tau = e^{-r(t-\tau)}\mathbb{E}^{\mathbb{P}^*}\left(\frac{\overline{P}_\tau(S_t)}{\mathbb{P}^*}\bigg|\mathcal{F}_\tau\right)$, where $(\mathcal{F}_\tau)_{\tau \in [0,T]}$ is the natural filtration. Similarly, consider an American option with payoff $\overline{P}_\tau$ and maturity $t \leq T$: the price of this option at time $\tau$ is

$$P_\tau = \sup_{s \in \mathcal{T}_{\tau,t}} \mathbb{E}^{\mathbb{P}^*}\left(e^{-r(s-\tau)}\overline{P}_\tau(S_s)\bigg|\mathcal{F}_\tau\right),$$

(1.1)

where $\mathcal{T}_{\tau,t}$ denotes the set of stopping times in $[\tau, t]$. The pricing model $\mathbb{P}^*$ must be compatible with the prices of the options observed on the market, whose number may be large. Model calibration consists of finding $\mathbb{P}^*$ such that the discounted price $(e^{-r\tau}S_\tau)_{\tau \in [0,T]}$ is a martingale, and s.t. the option prices computed by e.g. (1.1) in the case of American options coincide with the observed option prices. This is an inverse problem. We focus on the case when the observed prices $(\overline{P}_i)_i \in I$ are those of a family of American put options indexed by $i \in I$, with maturities $t_i$, (assuming for simplicity $T = \max_i t_i$) and strikes $x_i$.

The Black-Scholes model assumes that $(S_\tau)_{\tau \in [0,T]}$ is a geometric Brownian motion under $\mathbb{P}^*$: $dS_\tau = S_\tau(rdt + \sigma dW_\tau)$, where the volatility $\sigma$ is a constant. Unfortunately, this model is often too simple to match the observed option prices and must be replaced by more involved models:

1) Black-Scholes models with local volatility: the volatility is assumed to be a function of time and of the price of the underlying asset. This volatility function is calibrated by observing the option prices available on the markets and solving inverse problems involving either partial differential equations or inequalities, see[3, 7, 24] for volatility calibration with European options and [2, 5] with American options;

2) models where the volatility is also a stochastic process, see e.g. [21]. The option price is then found as a function of time, the price of the underlying asset and the volatility. These models also lead to parabolic partial differential equations or inequalities with possible degeneracies when the volatility vanishes; stochastic volatility calibration has been performed in [33];

3) models with Lévy driven underlying assets: Lévy processes are processes with stationary and independent increments which are continuous in probability, see the
The option price is found by solving partial integro-differential equations or inequalities. Calibration of Lévy models with European options has been discussed in [14, 15]. The present work is devoted to the calibration of Lévy processes with American options. At this stage, it is not yet necessary to discuss Lévy processes in detail. For the moment, we just assume that the model is characterized by parameters $\theta$ in a suitable class $\Theta$.

The last two classes of models describe incomplete markets: the knowledge of the historical price process alone does not allow to compute the option prices in a unique manner. When the option prices do not determine the model completely, additional information may be introduced into the problem by specifying a prior model. If the historical price process has been estimated statistically from the time series of the underlying asset, this knowledge has to be injected in the inverse problem; calling $P_0$ the prior probability measure obtained as an estimation of $P$, we are going to focus on least-square formulations of the type: find $\theta \in \Theta$ which minimizes

$$\sum_{i \in I} \omega_i \left( P^\theta_0(0, S_0, t_i, x_i) - p_i \right)^2 + \rho J_2(P^\theta_0, P_0),$$

(1.2)

where
- $\omega_i$ are suitable positive weights,
- $S_0$ is the price of the underlying asset today,
- $P^\theta_0(0, S_0, t_i, x_i)$ is the price of the option with maturity $t_i$ strike $x_i$, computed with the pricing model associated with $\theta$,
- $\rho J_2(P^\theta_0, P_0)$ is a regularization term which measures the closedness of the model $P^\theta_0$ to the prior. The number $\rho > 0$ is called the regularization parameter. This functional has two roles: 1) it stabilizes the inverse problem; for that, $\rho$ should be large enough and $J_2$ should be convex or at least convex in a large enough region; 2) it guarantees that $P^\theta$ remains close to $P_0$ in some sense. The choice of $J_2$ is very important: $J_2(P^\theta_0, P_0)$ is often chosen as the relative entropy of the pricing measure $P^\theta_0$ with respect to the prior model $P_0$, see [8], because the relative entropy becomes infinite if $P^\theta_0$ is not equivalent to $P_0$. Some authors have argued that such a choice may be too conservative in some cases, for two reasons: a) the historical data which determine the prior may be missing or partially available -b) in the context of e.g. volatility calibration, once the volatility is specified under $P_0$, then the volatility under $P^\theta_0$ must be the same for the relative entropy to be finite. In [9] a different approach was considered which allowed for volatility calibration.

Note that evaluating the functional in (1.2) requires solving #I linear complementarity problems (LCP for brevity) involving partial integro-differential (PID) operators in the variables $\tau$ and $S$, see §2.1 below. This approach was chosen in [2, 5] for calibrating local volatility with American options.

In the present case (Lévy driven assets), we show that there is a better approach which consists of computing the prices $P^\theta_0(0, S_0, t_i, x_i)$, $i \in I$, by using a single forward in time LCP with a PID operator in the variables maturity $t$ and strike $x$. This LCP is introduced in § 2.2, see (2.9-2.11) below. It is reminiscent of the forward equation (known as Dupire’s equation in the finance community) which is often used for local volatility calibration with vanilla European options, see [4, 18].

We then find a new least square problem, where the functional is evaluated by solving a single LCP involving a PID operator. The main goal of the paper is to study this least square problem theoretically for a rather general parameterization of the Lévy density $k$, see (2.14) below, with the volatility $\sigma$ bounded away from 0, and to give necessary optimality conditions. This problem has connections with some optimal control problems for variational inequalities studied in [11, 23, 32]. The article of Hintermüller [22] on an inverse problem for an elliptic variational inequality has inspired [2] and the present work.
As far as we know, this is the first attempt at calibrating Lévy processes with American options, so comparison with other methods is difficult. The results below can be used in practice because they have their discrete counterparts when finite elements or finite differences are used. The accuracy is expected to be similar to the one observed in [5].

The paper is organized as follows. In §2, we obtain the forward LCP (2.9-2.11) and make some assumptions on the Lévy density. In §3, we introduce a family of fractional weighted Sobolev spaces and give preliminary results on the nonlocal operator in (2.9). In §4 we carefully study the variational inequality stemming from (2.9)-(2.11). For the analysis, we must first study a regularized nonlinear problem posed in a bounded domain, then let the regularization parameter tend to 0 and the domain’s boundary tend to infinity. The sensitivity of the solution to variations of \( \sigma \) and \( k \) is discussed in §5. Finally, the inverse problem is studied in §6: necessary optimality conditions are given. Some technical proofs are postponed to §§7 and 8.

For the reader’s convenience, let us point out the main results of this work:

- The forward complementarity problem is written in (2.9)-(2.11) and the assumptions on \( k \) are described in § 2.3.
- Theorem 4.9 contains a result of existence and uniqueness for the variational inequality associated to (2.9)-(2.11) in suitable Sobolev spaces. It is also proved that the related free boundary stays in a bounded region. Note that by using the theory presented in [10], it is possible to study the variational inequality in Sobolev spaces with decaying weights as \( x \to 0 \) and \( x \to +\infty \), (actually the variable \( \log(x) \) was used instead of \( x \) in [10]). Here, we show that these weights can be avoided. Another advantage of the present analysis is that it can be extended to the case when \( \sigma = 0 \) by singular perturbation arguments, if the Lévy measure is chosen to keep the problem parabolic. This will be done in a forthcoming work, [1].
- The sensitivity of solutions w.r.t. variations of the Lévy process is studied in §5.
- Theorem 6.6 contains the necessary optimality conditions for the least square inverse problem. These conditions are obtained by first studying a modified inverse problem whose state variable satisfies the above mentioned regularized nonlinear problem, then by passing to the limit as the regularization parameter tends to zero.

### 2. Description of the model

#### 2.1. The backward linear complementarity problem. For a Lévy process \((X_\tau)_{\tau>0}\) on a filtered probability space, the Lévy-Khintchine formula says that there exists a function \( \chi : \mathbb{R} \to \mathbb{C} \) such that \( E(e^{iuX_\tau}) = e^{\tau \chi(u)} \), with

\[
\chi(u) = \frac{-\sigma^2 u^2}{2} + i\beta u + \int_{|z|\leq 1} (e^{iu z} - 1 - iuz) \nu(dz) + \int_{|z|> 1} (e^{iu z} - 1) \nu(dz),
\]

for \( \sigma \geq 0 \), \( \beta \in \mathbb{R} \) and a positive measure \( \nu \) on \( \mathbb{R} \setminus \{0\} \) such that \( \int_{\mathbb{R}} \min(1, z^2) \nu(dz) < +\infty \). The measure \( \nu \) is called the Lévy measure of \((X_\tau)_{\tau>0}\).

We assume that the discounted price of the risky asset is a martingale obtained as the exponential of a Lévy process: \( e^{-\tau r} S_\tau = S_0 e^{X_\tau} \). The fact that the discounted price is a martingale is equivalent to

\[
\int_{|z|> 1} e^z \nu(dz) < \infty, \quad \text{and} \quad \beta = -\frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^z - 1 - z1_{|z|\leq 1}) \nu(dz).
\]

We also assume that \( \int_{|z|> 1} e^{2z} \nu(dz) < \infty \), so the discounted price is a square integrable martingale.

In what follows, we assume that the Lévy measure has a density, \( \nu(dz) = k(z)dz \), with \( k \) possibly singular at \( z = 0 \). Doing so, we exclude the simplest Lévy processes obtained as the sum of Brownian motions and Poisson processes. This is not a
fundamental restriction in the sense that the methods proposed below could be extended (and even simplified) to calibrate the previously mentioned processes. The restriction is mainly done in order to focus on the difficulties posed by the possible singularities of $k$ at $z = 0$.

We note $\mathcal{B}$ the integral operator:

$$
(\mathcal{B}v)(S) = \int_{\mathbb{R}} \left( v(Se^z) - v(S) - S(e^z - 1)\frac{\partial}{\partial S}v(S) \right) k(z) dz.
$$

Consider an American option with payoff $T_\sigma$ and maturity $t$: in [10], Bensoussan and Lions assume $\sigma > 0$ and study the variational inequality stemming from the LCP: $P(t, S) = T_\sigma(S)$, and for $\tau < t$ and $S > 0$,

$$
\frac{\partial P}{\partial \tau}(\tau, S) + \frac{\sigma^2 S^2}{2} \frac{\partial^2 P}{\partial S^2}(\tau, S) + r S \frac{\partial P}{\partial S}(\tau, S) - r P(\tau, S) + (\mathcal{B}P)(\tau, S) \leq 0, \quad (2.1)
$$

$$
P(\tau, S) \geq T_\sigma(S), \quad (2.2)
$$

in suitable Sobolev spaces with decaying weights near $+\infty$ and 0, and prove that the price of the American option is $P_r = P(\tau, S_r)$. Other approaches with viscosity solutions are possible, see [35], especially in the case $\sigma = 0$. One advantage of the variational methods is that they provide stability estimates. For numerical methods for options on Lévy driven assets, see [4, 16, 17, 29, 30, 31].

### 2.2. The forward linear complementarity problem

As already explained, we aim at finding a forward LCP in the variables maturity/strike; a single solution of this problem will be needed for evaluating the cost function in (1.2).

Hereafter, since the observed prices are those of vanilla American put options, we use the notation

$$
P_\sigma(x) = (x - S)_+.
$$

If $T_\sigma(S) = (x - S)_+$, it can be seen that the solution of (2.1)-(2.3) is of the form

$$
P(\tau, S, t, x) = x g(\xi, y), \quad y = S/x \in \mathbb{R}_+, \xi = t - \tau \in (0, t), \quad (2.5)
$$

where $g$ is the solution of the complementarity problem independent of $x$, $g(0, y) = (1 - y)_+$ and for $0 < \xi \leq t$, $y \in \mathbb{R}_+$,

$$
-\frac{\partial g}{\partial \xi}(\xi, y) + \frac{\sigma^2 y^2}{2} \frac{\partial^2 g}{\partial y^2}(\xi, y) + ry \frac{\partial g}{\partial y}(\xi, y) - r g(\xi, y) + (B g)(\xi, y) \leq 0, \quad (2.6)
$$

$$
g(\xi, y) \geq (1 - y)_+, \quad (2.7)
$$

$$
\left( -\frac{\partial g}{\partial \xi}(\xi, y) + \frac{\sigma^2 y^2}{2} \frac{\partial^2 g}{\partial y^2}(\xi, y) + ry \frac{\partial g}{\partial y}(\xi, y) \right) \left( g(\xi, y) - (1 - y)_+ \right) = 0, \quad (2.8)
$$

where $(B v)(y) = \int_{\mathbb{R}} \left( v(ye^z) - v(y) - y(e^z - 1)\frac{\partial}{\partial y}v(y) \right) k(z) dz$. From this observation and the identities $x \frac{\partial g}{\partial \xi} = -\frac{\partial P}{\partial \tau}$, $x y \frac{\partial g}{\partial y} = -x \frac{\partial P}{\partial x} + P$, and $x y^2 \frac{\partial^2 g}{\partial y^2} = x^2 \frac{\partial^2 P}{\partial x^2}$, we deduce that, as a function of $t$ and $x$, $P(0, S, t, x)$ satisfies the following forward problem: $P(t = 0) = P_\sigma$ and for $t \in (0, T]$, and $x > 0$,

$$
\left( \frac{\partial P}{\partial t} - \frac{\sigma^2 x^2}{2} \frac{\partial^2 P}{\partial x^2} + r x \frac{\partial P}{\partial x} + B P \right) \geq 0, \quad (2.9)
$$

$$
P(t, x) \geq P_\sigma(x), \quad (2.10)
$$

$$
\left( \frac{\partial P}{\partial t} - \frac{\sigma^2 x^2}{2} \frac{\partial^2 P}{\partial x^2} + r x \frac{\partial P}{\partial x} + B P \right) (P - P_\sigma) = 0, \quad (2.11)
$$


where the integral operator \( B \) is defined by
\[
(Bu)(x) = -\int_{\mathbb{R}} k(z) \left( x(e^z - 1) \frac{\partial u}{\partial x}(x) + e^z(u(xe^{-z}) - u(x)) \right) \, dz. \tag{2.12}
\]

Note that the arguments yielding (2.9)-(2.11) are much easier than those used for getting Dupire’s equation, see [4, 18], because (2.5) does not hold with local volatility. Problem (2.9)-(2.11) can also be obtained by probabilistic arguments. Note also that finding a forward LCP in the variables \( t \) and \( x \) is not possible in the case of American options with local volatility, because the arguments in [4, 18] do not apply to nonlinear problems. This explains why, in [2, 5], the evaluation of the least square cost functional necessitates the solution of \#LCP instead of one here.

In this respect, we may say that with American options, the calibration of Lévy processes is easier than the calibration of local volatility.

### 2.3. Choice of the Lévy process.

We have already discussed our choice to take \( \nu(dz) = k(z)dz \), with
\[
\max \left( \int_{\mathbb{R}} \min(1, z^2) k(z)dz, \int_{1}^{+\infty} e^{2z} k(z)dz \right) < \infty. \tag{2.13}
\]

We need to make further restrictions on the Lévy process for several reasons:

1. in practice, we need to specify a class of Lévy densities \( k \) in order to define the inverse problem.
2. the analysis below will need problem (2.16-2.19) to be parabolic. This implies restrictions on the pair \((\sigma, k)\).

As it will appear in section 3.2 below, the restrictions in order to have a parabolic problem are

1. either \( \sigma > 0 \) and \( k \) satisfies (2.13),
2. or \( \sigma = 0 \) and \( k \) satisfies (2.13) and is sufficiently singular near \( z = 0 \); The result in § 3.2 will imply that choosing \( k(z) \sim |z|^{-1-2\alpha} \) with \( 1/2 < \alpha < 1 \) yields a parabolic problem. For keeping the length of this article reasonable, this case will be discussed elsewhere.

**Assumption 1.** For this reason, we assume that \( k \) is of the form
\[
k(z) = \psi(z)|z|^{-(1+2\alpha)}, \tag{2.14}
\]
where \( \psi \) is a nonnegative function in \( L^\infty(\mathbb{R}) \) such that \( \psi(z) \geq \psi > 0 \) in a fixed neighborhood of \( z = 0 \), and \( \alpha \) is s.t. \( -1/2 \leq \alpha < 1 \). We assume furthermore that (2.13) is satisfied. For practical purpose, one can impose further restrictions on \( \psi \), for example let \( \psi \) belong to a finite dimensional function space, but this needs not be discussed at this stage.

Assumption 1 holds for models of jump-diffusion type, for example Merton model (\( \sigma > 0 \) and the jumps in the log-price have a Gaussian distribution) or some Kou models (\( \sigma > 0 \) and the distribution of jumps is an asymmetric exponential with a fast enough decay at infinity), see [13], page 111. Indeed these models can be obtained by taking \( \alpha = -1/2 \) and choosing \( \psi \) properly. Assumption 1 also holds for some variance gamma processes (\( \sigma > 0, \alpha = 0 \)) and normal inverse Gaussian processes (\( \sigma > 0, \alpha = 1/2 \)), see [13], page 117, with a fast enough decay of the jump density at infinity. It also holds for some tempered stable processes, see [13], page 119, or some parabolic CGMY models discussed by Carr et al [12]. These last two models usually take \( \sigma = 0 \). Allowing \( \sigma > 0 \) in the analysis can be seen as a step toward \( \sigma = 0 \).

**Remark 1.** The assumption \( \psi(z) \geq \psi > 0 \) near 0 avoids ambiguities in the definition of the singularity of \( k \) at \( z = 0 \). It is a bit restrictive since for example a logarithmic singularity of \( k \) at \( z = 0 \) is ruled out. However, this assumption is unessential and most of the results below hold without it.
2.4. Change of unknown function in the forward problem. In order to have a datum with a compact support in \( x \), it is helpful to change the unknown function: we set

\[
u_0(x) = (S - x)_+; \quad u(t, x) = P(t, x) - x + S.\tag{2.15}\]

The function \( u \) satisfies: for \( t \in (0, T] \), and \( x > 0 \),

\[
\frac{\partial u}{\partial t} - \frac{\sigma^2 x^2}{2} \frac{\partial^2 u}{\partial x^2} + r x \frac{\partial u}{\partial x} + Bu \geq -r x, \tag{2.16}
\]

\[
u(t, x) \geq u_0(x), \tag{2.17}
\]

\[
\left( \frac{\partial u}{\partial t} - \frac{\sigma^2 x^2}{2} \frac{\partial^2 u}{\partial x^2} + r x \frac{\partial u}{\partial x} + Bu + r x \right) (u - u_0) = 0. \tag{2.18}
\]

The initial condition for \( u \) is

\[
u(t = 0, x) = u_0(x), \quad x > 0. \tag{2.19}\]

For writing the variational inequalities stemming from (2.16)-(2.19), we need to introduce suitable weighted Sobolev spaces. In particular, fractional order weighted Sobolev spaces will be useful for studying the nonlocal part of the operator.

3. Preliminary results.

3.1. Functional setting.

3.1.1. Sobolev spaces on \( \mathbb{R} \). For a real number \( s \), let the Sobolev space \( H^s(\mathbb{R}) \) be defined as follows: the distribution \( w \) defined on \( \mathbb{R} \) belongs to \( H^s(\mathbb{R}) \) if and only if its Fourier transform \( \hat{w} \) satisfies

\[
\int_{\mathbb{R}} (1 + \xi^2)^s |\hat{w}(\xi)|^2 d\xi < +\infty.
\]

The spaces \( H^s(\mathbb{R}) \) are Hilbert spaces, with the inner product and norm:

\[
(w_1, w_2)_{H^s(\mathbb{R})} = \int_{\mathbb{R}} (1 + \xi^2)^s \overline{\hat{w}_1(\xi)} \hat{w}_2(\xi) d\xi, \quad \|w\|_{H^s(\mathbb{R})} = \sqrt{(w, w)_{H^s(\mathbb{R})}}.
\]

For two real numbers \( s_1, s_2, s_1 \leq s_2 \), \( H^{s_2}(\mathbb{R}) \subset H^{s_1}(\mathbb{R}) \) with a continuous injection. It can be seen that \( H^0(\mathbb{R}) = L^2(\mathbb{R}) \) and that if \( s \) is a positive integer, \( H^s(\mathbb{R}) \) is the space of all the functions whose derivatives up to order \( s \) are square integrable. If \( s \) is a nonnegative integer, the norm \( \|\cdot\|_{H^s(\mathbb{R})} \) is equivalent to the norm \( v \mapsto \sqrt{\sum_{\ell=0}^m \| \frac{d^\ell v}{dy^\ell} \|_{L^2(\mathbb{R})}^2} \). If \( s = 0 \) or not an integer, the norm \( \|\cdot\|_{H^s(\mathbb{R})} \) is equivalent to

\[
v \mapsto \sqrt{\sum_{\ell=0}^m \| \frac{d^\ell v}{dy^\ell} \|_{L^2(\mathbb{R})}^2 + \int_{\mathbb{R}} \int_{\mathbb{R}} |y - z|^{2(m-s) - 1} \left( \frac{d^m v}{dy^m}(y) - \frac{d^m v}{dy^m}(z) \right)^2 dy dz}, \tag{3.1}
\]

where \( m \) is the integer part of \( s \). For \( s \geq 0 \), the space \( \mathcal{D}(\mathbb{R}) \) is dense in \( H^s(\mathbb{R}) \).

It is well known (see [27, 6]) that if \( 0 < s < 1 \), then \( H^s(\mathbb{R}) \) can be obtained by real or complex interpolation between the spaces \( H^1(\mathbb{R}) \) and \( L^2(\mathbb{R}) \) (the parameter for the real interpolation is \( \nu = 1/2 - s \), see [6] page 204), and that the norm obtained by the interpolation process is equivalent to the one defined in (3.1).

For \( s \geq 0 \), \( H^{-s}(\mathbb{R}) \) is the dual of \( H^s(\mathbb{R}) \), and for \( s > 0 \), the norm \( \|\cdot\|_{H^{-s}(\mathbb{R})} \) is equivalent to the norm \( v \mapsto \sup_{w \in H^s(\mathbb{R}), w \neq 0} \frac{|(v, w)|}{\|w\|_{H^s(\mathbb{R})}} \). If \( s \) is a nonnegative integer, we define the semi-norm \( |v|_{H^s(\mathbb{R})} = \sqrt{\sum_{\ell=0}^m \| \frac{d^\ell v}{dy^\ell} \|_{L^2(\mathbb{R})}^2} \). If \( s > 0 \) is not an integer, we define \( |v|_{H^s(\mathbb{R})} \) by

\[
|v|_{H^s(\mathbb{R})}^2 = \sum_{\ell=0}^m \| \frac{d^\ell v}{dy^\ell} \|_{L^2(\mathbb{R})}^2 + \int_{\mathbb{R}} \int_{\mathbb{R}} \left( \frac{d^m v}{dy^m}(y) - \frac{d^m v}{dy^m}(z) \right)^2 |y - z|^{2m} dy dz, \]

where \( m \) is the integer part of \( s \).
3.1.2. Some weighted Sobolev spaces on $\mathbb{R}_+$. Let $V^1$ be the weighted Sobolev space

$$V^1 = \left\{ v \in L^2(\mathbb{R}_+), x^s \frac{\partial v}{\partial x} \in L^2(\mathbb{R}_+) \right\},$$

which is a Hilbert space with the norm $\|v\|_{V^1} = \sqrt{\|v\|_{L^2(\mathbb{R}_+)}^2 + \|x^s \frac{\partial v}{\partial x}\|_{L^2(\mathbb{R}_+)}^2}$. It is proved in [4] that $D(\mathbb{R}_+)$ is a dense subspace of $V^1$, and that the following Poincaré inequality is true: for all $v \in V^1$,

$$\|v\|_{L^2(\mathbb{R}_+)} \leq 2 \|x \frac{\partial v}{\partial x}\|_{L^2(\mathbb{R}_+)}. \quad (3.2)$$

Therefore, the semi-norm $|.|_{V^1}$: $|v|_{V^1} = \|x \frac{\partial v}{\partial x}\|_{L^2(\mathbb{R}_+)}$ is a norm equivalent to $|.|_{V^1}$.

For a function $v$ defined on $\mathbb{R}_+$, call $\check{v}$ the function defined on $\mathbb{R}$ by

$$\check{v}(y) = v(\exp(y)) \exp(y/2). \quad (3.3)$$

By using the change of variable $y = \log(x)$, it can be seen that the mapping $v \mapsto \check{v}$ is a topological isomorphism from $L^2(\mathbb{R}_+)$ onto $L^2(\mathbb{R})$, and from $V^1$ onto $H^1(\mathbb{R})$. This leads to defining the space $V^s$, for $s \in \mathbb{R}$, by

$$V^s = \{ v : \check{v} \in H^s(\mathbb{R}) \},$$

which is a Hilbert space with the norm $\|v\|_{V^s} = \|\check{v}\|_{H^s(\mathbb{R})}$. Using the interpolation theorem given e.g. in [6] Theorem 7.17, one can prove that if $0 < s < 1$, then $V^s$ can be obtained by real interpolation between the spaces $V^1$ and $L^2(\mathbb{R}_+)$ (the parameter for the real interpolation is $\nu = 1/2 - s$), and that the norm obtained by the interpolation process is equivalent to the one defined above.

For $s > 0$, the space $V^{-s}$ is the topological dual of $V^s$.

**Lemma 3.1.** Let $s$ be a real number such that $1/2 < s \leq 1$. Then for all $v \in V^s$, $v$ is continuous on $(0, +\infty)$ and there exists a constant $C > 0$ such that

$$\sqrt{x}|v(x)| \leq C\|v\|_{V^s}, \quad \forall x \in [1, +\infty). \quad (3.4)$$

**Proof.** From the Sobolev continuous imbedding $H^s(\mathbb{R}) \subset L^\infty(\mathbb{R}) \cap C^0(\mathbb{R})$ for $s > 1/2$, we see that $V^1 \subset C^0((0, +\infty))$ and there exists a constant $C$ such that $|v(x)| = |\check{v}(\log(x))|/\sqrt{x} \leq C\|v\|_{H^s(\mathbb{R})}/\sqrt{x} = C\|v\|_{V^s}/\sqrt{x}, \forall v \in V^s, \forall x \geq 1$. $\square$

For a continuous and nonnegative function $\phi$ defined on $\mathbb{R}$, and a measurable function $v$ on $\mathbb{R}_+$, consider

$$|v|_{\phi,s}^2 = \int_{\mathbb{R}_+} dx \int_{\mathbb{R}} \frac{\phi(z)}{|z|^{1+2s}} (v(xe^{-z}) - v(x))^2 \, dz, \quad \text{and} \quad \|v\|_{\phi,s} = \sqrt{|v|_{\phi,s}^2 + \|v\|_{L^2(\mathbb{R}_+)}^2}.$$

**Lemma 3.2.** Let $\phi$ be a continuous and nonnegative function defined on $\mathbb{R}$. If $\phi(0) > 0$ and if the function $z \mapsto \phi(z) \max(e^z, 1)$ is bounded, then for any $s \in (0, 1)$, $\|v\|_{\phi,s}$ is a norm on $V^s$ equivalent to the norm $|.|_{V^s}$.

**Proof.** For the reader’s ease, the proof is postponed to § x7. $\square$

**Remark 2.** Lemma 3.2 remains true if $\phi$ is a function in $L^\infty(\mathbb{R})$ and if for a given positive constant $\phi \geq \phi > 0$ a.e. in a neighborhood of 0.

**Remark 3.** If the assumption $\phi(0) > 0$ is not satisfied, then the conclusion of Lemma 3.2 becomes: $\exists C > 0$ such that $|u|_{\phi,s} \leq C\|u\|_{V^s}$, $\forall u \in V^s$.

3.2. The integro-differential operator.
3.2.1. The integral operator. We study the operator $B$ defined in (2.12).

**Lemma 3.3.** Let $(\alpha, \psi)$ satisfy Assumption 1. For each $s \in \mathbb{R}$,
- if $\alpha > 1/2$, then the operator $B$ is continuous from $V^s$ to $V^{s-2\alpha}$,
- if $\alpha < 1/2$, then the operator $B$ is continuous from $V^s$ to $V^{s-1}$,
- if $\alpha = 1/2$, then the operator $B$ is continuous from $V^s$ to $V^{s-1-\epsilon}$, for any $\epsilon > 0$.

**Proof.** See §7.

**Corollary 3.4.** If $(\alpha, \psi)$ satisfy Assumption 1 and if $1/2 < \alpha < 1$, then the operator $B$ is continuous from $V^\alpha$ to $V^{-\alpha}$.

**Lemma 3.5.** If $(\alpha, \psi)$ satisfy Assumption 1 and $1/2 < \alpha < 1$, then $\forall v, w \in V^\alpha$,

$$
(Bu, v) + (Bv, u) = \left\{ \int_{\mathbb{R}^+} \int_{\mathbb{R}} k(z)e^{\alpha}(u(x) - u(xe^{-z}))(v(x) - v(xe^{-z}))dx dz \\
+ \int_{\mathbb{R}} k(z)(2e^z - e^{2z} - 1)dz \right\} \int_{\mathbb{R}^+} u(x)v(x)dx
$$

where $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $V^\alpha$ and $V^\alpha$.

If $-1/2 \leq \alpha \leq 1/2$, then, (3.5) is true for $u, v \in V^s$, $s > 1/2$, defining $\langle \cdot, \cdot \rangle$ as the duality pairing between $V^{-s}$ and $V^s$.

**Proof.** See §7.

**Remark 4.** If $(\alpha, \psi)$ satisfy Assumption 1, the operator $B^T$ defined by

$$
B^T u(x) = \int_{\mathbb{R}} k(z) \left( x(e^z - 1)\frac{\partial u}{\partial x}(x) - e^{2z}u(xe^z) + (2e^z - 1)u(x) \right) dz
$$

is a continuous operator from $V^s$ to $V^{s-2\alpha}$, if $\alpha > 1/2$,
- is a continuous operator from $V^s$ to $V^{s-1}$, if $\alpha < 1/2$.
- is a continuous operator from $V^s$ to $V^{s-1-\epsilon}$, for any $\epsilon > 0$, if $\alpha = 1/2$.

If $\alpha > 1/2$, then $\forall u, v \in V^\alpha$, $\langle B^T u, v \rangle = \langle Bu, v \rangle$. This identity holds for all $u, v \in V^s$ with $s > 1/2$ if $\alpha = 1/2$.

**Lemma 3.6.** If $(\alpha, \psi)$ satisfy Assumption 1 and if
- either $\alpha < 1/2$,
- or $\psi$ is continuous near 0 and there exists a bounded function $\omega : \mathbb{R} \rightarrow \mathbb{R}$ and two positive numbers $\zeta$ and $C$ such that $\psi(z)e^{3/2z} - \psi(0)e^{-3/2z} = z\omega(z)$, with $|\omega(z)| \leq C|z|e^{-\zeta|z|}$, for all $z \in \mathbb{R}$,

then for any $s \in \mathbb{R}$, the operator $B - B^T$ is continuous from $V^s$ to $V^{s-1}$.

**Proof.** See §7.

**Proposition 3.7 (Gårding inequality).** Let $(\alpha, \psi)$ satisfy Assumption 1. If $1/2 < \alpha < 1$, there exist two constants $C > 0$ and $\lambda \geq 0$ such that, $\forall v \in V^\alpha$,

$$
\langle Bu, v \rangle \geq C \|u\|^2_{L^2(\mathbb{R}^+)} - \lambda \|v\|^2_{L^2(\mathbb{R}^+)}
$$

If $\alpha \leq 1/2$, then (3.7) holds for any $v \in V^s$, $s > 1/2$ ($\langle \cdot, \cdot \rangle$ standing for the duality pairing between $V^{-s}$ and $V^s$), with $C = 0$ if $\alpha < 0$.

**Proof.** If $0 < \alpha < 1$, the function $\phi : z \mapsto e^{\alpha}(\psi(z)$ satisfies the assumptions of Remark 2. Therefore, $u \mapsto \sqrt{\|u\|^2_{L^2(\mathbb{R}^+)} + \int_{\mathbb{R}} \int_{\mathbb{R}} k(z)e^{\alpha}(u(x) - u(xe^{-z}))^2dx dz}$ is a norm on $V^\alpha$ equivalent to the norm $\|\cdot\|_{V^\alpha}$. From this and (3.5), we deduce (3.7).

Consider the two situations
- $1/2 < \alpha < 1$, $\psi$ and $u \in V^\alpha$: it can be shown (using the interpolation theorem 7.17 in [6]) that the functions $u_+$ and $u_-$ belong to $V^\alpha$;
- $\alpha \leq 1/2$ and $u \in V^s$, $s > 1/2$.

In both cases, $\int_{\mathbb{R}} \int_{\mathbb{R}} k(z)e^{\alpha}u_-(xe^{-z})u_+(x)dx dz$ is well defined because

$$
\int_{\mathbb{R}} \int_{\mathbb{R}} k(z)e^{\alpha}u_-(xe^{-z})u_+(x)dx dz = \left[ \int_{\mathbb{R}} k(z)e^{\alpha}(u_-(xe^{-z}) - u_-(x))u_+(x)dx \right] dz
$$

$$
\leq C \|u_+\|_{L^2(\mathbb{R}^+)} \left( \int_{\mathbb{R}} \int_{\mathbb{R}} k(z)e^{\alpha}(u_-(xe^{-z}) - u_-(x))^2dx dz \right)^{1/2}
$$
and is nonnegative. Therefore,

\[ \langle Bu, u_+ \rangle = \langle Bu_+, u_+ \rangle - \int_{\mathbb{R}^+} \int_{\mathbb{R}} k(z) e^z (u(xe^{-z}) - u_+(xe^{-z}))u_+(x)dxdz \]

\[ = \langle Bu_+, u_+ \rangle + \int_{\mathbb{R}^+} \int_{\mathbb{R}} k(z) e^z u_-(xe^{-z})u_+(x)dxdz \geq \langle Bu_+, u_+ \rangle. \]

We have proved the

**Lemma 3.8.** If \((\alpha, \psi)\) satisfy Assumption 1 then there exist two constants \(C > 0\) and \(\lambda \geq 0\) such that, for all \(u \in V^\alpha\) if \(\alpha > 1/2\) or for all \(u \in V^s\) \(s > 1/2\) if \(\alpha \leq 1/2\),

\[ \langle Bu, u_+ \rangle \geq C\|u_+\|^2_{V^\alpha} - \lambda\|u_+\|^2_{L^2(\mathbb{R}^+)}, \]

with \(C = 0\) if \(\alpha < 0\).

**3.2.2. The integro-differential operator.** With \(B\) defined in (2.12), we introduce the integro-differential operator \(A:\)

\[ Av = -\frac{\sigma^2 x^2}{2} \frac{\partial^2 v}{\partial x^2} + rx \frac{\partial v}{\partial x} + Bv, \] \[ (3.9) \]

where \(\sigma\) and \(r\) are nonnegative real numbers. In this work, we limit ourselves to the case \(\sigma > 0\). The case \(\sigma = 0\), \(\alpha > 1/2\) requires working in the fractional Sobolev spaces described above and will be treated in [1]. Since the space \(V^1\) will play a special role, we use the shorter notation \(V = V^1\).

If \(\sigma > 0\), and if \((\alpha, \psi)\) satisfy Assumption 1, then

- \(A\) is a continuous operator from \(V\) to \(V^{-1}\),
- we have the Gårding inequalities: there exist \(\underline{c} > 0\) and \(\lambda \geq 0\) such that
  \[ \langle Av, v \rangle \geq \underline{c}\|v\|^2_V - \lambda\|v\|^2_{L^2(\mathbb{R}^+)}, \] \[ \forall v \in V, \]

\[ \langle Av, v_+ \rangle \geq \underline{c}\|v_+\|^2_V - \lambda\|v_+\|^2_{L^2(\mathbb{R}^+)}, \] \[ \forall v \in V. \]

- The operator \(A + \lambda I\) is one to one and continuous from \(V^2\) onto \(L^2(\mathbb{R}^+)\), with a continuous inverse.

**Remark 5.** The assumption that \(\psi > 0\) near \(z = 0\) is not necessary for \(A\) to have the above properties. Its role is to allow a clear identification of the kernel’s singularity at \(z = 0\).

**3.3. The variational inequalities.** We are ready to write the variational inequalities corresponding to the LCP (2.16)-(2.19).

We introduce the closed subspace of \(V:\)

\[ K = \{v \in V, \ v(x) \geq u_0(x) \ \text{in} \ \mathbb{R}^+\}. \] \[ (3.12) \]

The variational problem will consist of looking for \(u \in L^2(0, T; V) \cap C^0([0, T]; L^2(\mathbb{R}^+))\), with \(\frac{\partial u}{\partial t} \in L^2((0, T) \times \mathbb{R}^+)\), such that

1. there exists a constant \(X_T > S\) such that \(u(t, x) = 0, \forall t \in [0, T], \forall x \geq X_T\),
2. \(u(t) \in K\) for almost every \(t \in (0, T)\),
3. For almost every \(t \in (0, T)\), for any \(v \in K\) with bounded support,

\[ \left\langle \frac{\partial u}{\partial t} + Au + rx, v - u \right\rangle \geq 0, \] \[ (3.13) \]

where \(\langle \cdot, \cdot \rangle\) stands for the duality pairing between \(V^\prime\) (the dual of \(V\)) and \(V\).

4. \(u(t = 0) = u_0\).

Hereafter, this problem will be referred to as (VIP). The goal of section 4 below is to prove that (VIP) has a unique solution and to study its properties.
4. Analysis of the variational inequalities.

4.1. Orientation. Hereafter, we assume that \( \sigma > 0 \).

Problem (2.16)-(2.19) is posed in an unbounded domain. This is a technical difficulty in order to use variational methods, and we first have to replace this problem by a similar one posed in a bounded domain. Therefore, the program is to

1. approximate (2.16)-(2.19) by a similar problem posed in \([0, T] \times [0, X]\), for some given positive parameter \( X > S \), and write the related variational problem, which will be called (VIP) below;
2. solve first a penalized version of (VIP) by introducing a semilinear monotone operator. Pass to the limit as the penalty parameter tends to zero;
3. prove that the free boundary of (VIP) stays in a bounded domain as \( X \) tends to infinity: this will show that for \( X \) large enough a solution of (VIP) is actually a solution of (VIP);
4. obtain estimates for the solution of (VIP) independent of the parameters \( (\sigma, \alpha, \psi) \), when these parameters vary in a suitably defined set.

4.2. Approximation of (VIP) in a bounded domain. Let \( X \) be a positive number greater than \( S \). Hereafter, for a function \( v \in L^2((0, X)) \) we call \( \mathcal{E}_X(v) \) the function in \( L^2(\mathbb{R}_+) \) obtained by extending \( v \) by 0 outside \((0, X)\). We introduce the Sobolev spaces \( W^1_X = \{ v \in L^2((0, X)), x \frac{\partial v}{\partial x} \in L^2((0, X)) \} \) and \( W^2_X = \{ v \in W^1_X, x \frac{\partial^2 v}{\partial x^2} \in L^2((0, X)) \} \). For \( \beta \), \( 0 < \beta < 1 \), \( W^\beta_X \) is the space obtained by real interpolation between \( W^1_X \) and \( L^2(0, X) \) with parameter \( \nu = 1/2 - \beta \), see \([6]\) page 204,\(27\)), and \( W^{1+\beta} \) is the dual of \( W^\beta_X \). Thanks to Lemma 3.1, we know that for \( \beta > 1/2 \), a function \( v \in V^\beta_X \) is continuous in \([0, X]\) and vanishes at \( X \).

Since the space \( V^\beta_X \) will often be used, we introduce the special notations

\[
V^\beta_X = \{ v \in L^2(0, X), x \frac{\partial v}{\partial x} \in L^2((0, X)) \}
\]

and \( \|v\|_{V^\beta_X} = \|\mathcal{E}_X(v)\|_\beta \). We define the operators \( A_X \) and \( B_X \), \( V^\beta_X \to V^{\beta'}_X \),

\[
(A_X v, w) = \langle A \mathcal{E}_X(v), \mathcal{E}_X(w) \rangle \quad \text{and} \quad (B_X v, w) = \langle B \mathcal{E}_X(v), \mathcal{E}_X(w) \rangle.
\]

A Gårding inequality for \( A_X \) is deduced from (3.10), with constants independent of \( X \). We define

\[
D_X = \{ v \in V^\beta_X : A_X v \in L^2((0, X)) \}.
\]

It follows from the Gårding inequality that \( (A_X, D_X) \) is the infinitesimal generator of an analytic semi-group \([34]\). Proposition 4.1 below contains information on \( D_X \):

**Proposition 4.1.** If \( v \in D_X \), then for any number \( X' < X \), \( v|_{(0, X')} \in W^\beta_{X'} \).

For \( \alpha \), \( 0 < \alpha < 3/4 \), \( D_X = W^\beta_X \cap V_X \). For \( \alpha \), \( 3/4 < \alpha < 1 \), there exists \( \epsilon > 0 \) such that \( D_X \subset W^{3/2+\epsilon}_X \cap V_X \). In any case, if \( v \in D_X \), then \( \frac{\partial v}{\partial x} \in C^\alpha((0, X)) \).

**Proof.** See § 8. \( \square \)

**Remark 6.** It can be proved by lengthy calculations that if \( v(x) = X - x \), \((\text{note that } v \in W^\beta_X \cap V_X)\) then \( A_X v \) behaves like \((X - x)^{1-2\alpha} \) near \( x = X \), so \( A_X v \in L^2((0, X)) \) if \( \alpha > 3/4 \).

We introduce

\[
K_X = \{ v \in V_X : v(x) \geq u_0(x) \ \text{in} \ (0, X) \}.
\]

We are going to look for \( u_X \in L^2(0, T; V_X) \cap C^0([0, T]; L^2((0, X))) \), with \( \frac{\partial u_X}{\partial t} \in L^2((0, T) \times (0, X)) \), such that
1. \( u_X(t) \in K_X \) for almost every \( t \in (0, T) \).

2. For almost every \( t \in (0, T) \),

\[
\left\langle \frac{\partial u_X}{\partial t} + A_X u_X + r x, v - u_X \right\rangle \geq 0,
\]

for any \( v \in K_X \). Here \( \langle \cdot, \cdot \rangle \) stands for the duality pairing between \( V_X^* \) (the dual of \( V_X \)) and \( V_X \).

3. \( E_X(u_X)(t = 0) = u_0 \).

Hereafter, this problem will be referred to as (VIP\(_X\)). In order to prove that (VIP\(_X\)) has a unique solution, we follow [25] and introduce first a sequence of monotone problems which can be seen as penalized versions of (4.5): find \( u_{X, \epsilon} \) such that

\[
\frac{\partial u_{x, \epsilon}}{\partial t} + A_X u_{x, \epsilon} + r x (1 - 1_{\{x > S\}}) \mathcal{V}_\epsilon (u_{x, \epsilon}) = 0, \quad t \in (0, T], \quad 0 < x < X,
\]

\[
u_{x, \epsilon}(t = 0, x) = u_0(x), \quad 0 < x < X,
\]

\[
u_{x, \epsilon}(t, x) = 0, \quad t \in (0, T],
\]

where \( \mathcal{V}_\epsilon (u) = \mathcal{V}(u/\epsilon) \) and \( \mathcal{V} \) is a smooth nonincreasing convex function such that

\[
\mathcal{V}(0) = 1, \quad \mathcal{V}(u) = 0 \quad \text{for} \quad u \geq 1, \quad 0 \geq \mathcal{V}'(u) \geq -2 \quad \text{for} \quad 0 \leq u \leq 1.
\]

In what follows, we call \( u^{(E)}_X \) and \( u^{(E)}_{\infty} \) the solutions to the linear problems:

\[
\frac{\partial u^{(E)}_X}{\partial t} + A_X u^{(E)}_X = 0, \quad \frac{\partial u^{(E)}_{\infty}}{\partial t} + A_X u^{(E)}_{\infty} = -r x, \quad t \in (0, T], \quad 0 < x < X,
\]

\[
u^{(E)}_X (t = 0, x) = u_0(x), \quad 0 < x < X,
\]

\[
u^{(E)}_{\infty} (t, X) = u^{(E)}_X (t, X) = 0 \quad t \in (0, T],
\]

It can be seen that \( u^{(E)}_X (t, 0) = S, \forall t \in [0, T] \) and that

\[
u^{(E)}_X (t, x) > S - x, \quad \forall (t, x) \in (0, T] \times (0, X].
\]

Let \( u^{(E)} \) be the solution of the linear problem:

\[
\frac{\partial u^{(E)}}{\partial t} + A u^{(E)} = 0, \quad t \in (0, T], \quad x > 0, \quad u^{(E)}(t = 0, x) = u_0(x), \quad x > 0.
\]

The function \( u^{(E)} \) is smooth near \( x = 0 \) and \( \frac{\partial u^{(E)}}{\partial x}(t, 0) = -1, \forall t \geq 0 \).

**Theorem 4.2.** If \((\alpha, \psi)\) satisfy Assumption 1 and if \( \sigma > 0 \), then (4.6) has a unique weak solution \( u_{x, \epsilon} \in L^2(0, T; V_X) \cap C^0([0, T]; L^2(0, X)) \). It satisfies

\[
u^{(E)}_X \leq u_{x, \epsilon} \leq u^{(E)}_X \leq u^{(E)}.
\]

The function \( u_{x, \epsilon} \) belongs to \( C^0([0, T]; K_X) \cap L^2(0, T; D_X) \) and is continuous and nondecreasing w.r.t. \( t \). For two positive numbers \( \epsilon' < \epsilon \), we have

\[
u_{x, \epsilon'} \leq u_{x, \epsilon} \leq u_{x, \epsilon} + \epsilon.
\]

The quantities \( \| \nu_{x, \epsilon} \|_{L^\infty(0, T; V_X)} \), \( \| u_{x, \epsilon} \|_{L^2(0, T; D_X)} \), \( \| \frac{\partial u_{x, \epsilon}}{\partial t} \|_{L^2(0, T; \times \langle 0, X \rangle)} \) are bounded independently of \( \epsilon \). The quantities \( \| \nu_{x, \epsilon} \|_{L^\infty(0, T; L^2(0, X))} \) and \( \| u_{x, \epsilon} \|_{L^2(0, T; V_X)} \) are bounded independently of \( X \).

**Proof.** See §8. \( \square \)

**Theorem 4.3.** The function \( x \frac{\partial u_{x, \epsilon}}{\partial x} \) is the sum of \( \hat{z}_{x, \epsilon} \in C^0([0, T]; L^2(0, X)) \) and of \( \hat{z}_{x, \epsilon} \in L^2(0, T; V_X) \) such that \( \hat{z}_{x, \epsilon} \leq 0 \) and \( \lim_{\epsilon \to 0} \| \hat{z}_{x, \epsilon} \|_{L^2(0, T; V_X)} = 0 \). Finally, for two numbers \( X \) and \( X' \) such that \( S < X < X' \), for any \( \epsilon > 0 \),

\[
E_X(u_{x, \epsilon}) \leq E_{X'}(u_{x, \epsilon}).
\]
The function $u_X$ is continuous in $[0, T] \times [0, X]$, with $u_X(t, 0) = S, \forall t \in [0, T]$, $u_X^E \leq u_X \leq u_X^E$, and $u_X(t, x) > u_o(x)$ for $0 < t \leq T$ and $0 < x \leq S$. The function $u_X$ is nondecreasing with respect to $t$, and nonincreasing with respect to $x$. The quantities $\|E_X(u_X)\|_{L^2(0, T; L^2(\mathbb{R}_+))}$ and $\|E_X(u_X)\|_{L^2(0, T; V)}$ are bounded independently of $X$.

For $\epsilon > 0$, we have the bounds

$$u_X \leq u_X, \epsilon \leq u_X + \epsilon,$$

(4.11)

and the sequence $u_{X, \epsilon}$ converges to $u_X$ uniformly as $\epsilon \to 0$.

Proof. The proof mainly consists of passing to the limit in (4.6) as $\epsilon \to 0$. It uses the Minty trick, see [25]. We skip it since it is rather classical. 

Lemma 4.5. If $(\alpha, \psi)$ satisfy Assumption 1 and if $\sigma > 0$, there exists a nondecreasing function $\gamma_X : (0, T] \to (S, X)$, such that the set $\{(t, x) : u_X(t, x) = u_o(x)\}$ coincides with the set $\{(t, x) : x \geq \gamma_X(t)\}$. Calling

$$\mu_X = \frac{\partial u_X}{\partial t} + A_X u_X + rx,$$

(4.12)

we have a.e.

$$0 \leq \mu_X \leq rx 1_{\{u_X = 0\}} = rx 1_{\{x \geq \gamma_X(t)\}}.$$

(4.13)

Proof. We know that for all $t \in [0, T]$, $u_X(t, X) = u_o(X) = 0$. Thus, at each time $t$, the set where $u_X(t, x)$ coincides with $u_o$ is nonempty. It is closed since $u_X$ and $u_o$ are continuous. We also know that $u_X(t, x) > u_o(x)$ for $t > 0$ and $x \leq S$; thus, $\{x > 0 \text{ s.t. } u_X(t, x) = u_o(x)\} \subset (S, X)$ for $t > 0$. On the other hand, for all $t \in (0, T]$, the function $u_X(t)$ is nonincreasing with respect to $x$, so $\{x > 0 \text{ s.t. } u_X(t, x) = u_o(x)\}$ is an interval $[\gamma_X(t), X]$, with $\gamma_X(t) > S$. Since $u_X$ is nondecreasing with respect to $t$, the function $\gamma_X$ is nondecreasing. With $\mu_X \in L^2((0, T) \times \mathbb{R}_+)$ given by (4.12), we have $\mu_X = 0$ a.e. in the open region where $u_X > 0$. Now, $\mu_X$ is the weak limit of $rx 1_{x > S} V(u_{X, \epsilon})$ in $L^2((0, T) \times (0, X))$. From (4.11), we deduce that $rx 1_{x > S} V(u_{X, \epsilon}) \leq rx 1_{x > S} V(u_X)$, and $1_{x > S} V(u_X)$ converges pointwise to $1_{\{u_X = 0\}}$. Therefore, $\mu_X \leq rx 1_{\{u_X = 0\}}$.

Proposition 4.6. If $(\alpha, \psi)$ satisfy Assumption 1 and if $\sigma > 0$, the function $\gamma_X$ is nondecreasing and lower semi-continuous. The graph of $\gamma_X$ has measure 0 (Lebesgue measure in $\mathbb{R}^2$) and

$$\mu_X(t, x) = 1_{\{u_X(t, x) = 0\}}(rx + B_X u_X(t, x))$$

$$= 1_{\{u_X(t, x) = 0\}}(rx - \int_{\mathbb{R}} k(z)e^z u_X(t, x e^{-z})dz)$$

(4.14)

for a.a. $t, x$.

Proof. We have already seen that $\gamma_X$ is nondecreasing. The epigraph of $\gamma_X$ is the set where $u_X$ vanishes. This region is closed since $u_X$ is continuous. Since $\gamma_X$ has a left and right limit at each point $t$, the graph of $\gamma_X$ has measure 0 (Lebesgue measure in $\mathbb{R}^2$), see Theorem 3.7 in [2] for the proof. As a consequence, the boundary of the coincidence set $\{u_X = 0\}$ has measure 0 (Lebesgue measure in $\mathbb{R}^2$). From this and since the identity $\mu_X(t, x) = rx - \int_{\mathbb{R}} k(z)e^z u_X(t, x e^{-z})dz$ is true in the set $\{x > \gamma_X(t)\}$, we obtain (4.14).
Remark 7. We will not try to obtain further regularity results on \( \gamma_X \). Yet, this is certainly an interesting topic on which little seems to be known. Let \( T_X \) be defined by
\[
T_X = \sup \{ t, \ 0 < t \leq T, \ \gamma_X(t) < X \}. \tag{4.15}
\]

Since \( \gamma_X \) is nondecreasing, we know that if \( T_X < T \), then \( \forall t \in [T_X, T], \gamma_X(t) = X \). Note that \( \mathcal{E}_X(u_X) \) is a solution of (2.16)-(2.19) in \( (0, T_X) \times \mathbb{R}_+ \), so for all \( X' > X, \) \( \mathcal{E}_X(u_X) \) coincides with \( \mathcal{E}_{X'}(u_{X'}) \) for \( 0 < t < T_X \). In particular, this implies that \( X \mapsto T_X \) is a nondecreasing function.

**Lemma 4.7.** If \((\alpha, \psi)\) satisfy Assumption 1 and if \( \sigma > 0 \), there exists \( X_T > S \) such that for all \( X \geq X_T, \) \( T_X = T \). For \( X > X_T, \) \( u_X \in L^2(0, T; W_X) \).

**Proof.** The proof is done by contradiction: if \( X_T \) does not exist, then \( \lim_{X \to \infty} T_X = T < T \). We have \( \frac{\partial u}{\partial t} + A_X u_X = -rx \) in \([T, T] \times (0, X)\), for all \( X > S \). We choose a smooth and nonnegative function \( \phi \) defined on \( \mathbb{R} \) with compact support, and for \( y > 0 \), we call \( \phi_y \) the function \( \phi_y(x) = \phi(x - y)/\sqrt{y} \). Then we take \( \phi_y \) as a test function in (4.12). We have
\[
\int_0^X (u_X(t, x) - u_X(T, x))\phi_y(x) + \int_T^T (A_X u_X(t, \phi_y))dt = -r \int_T^T \int_0^X x\phi_y(x)dx.
\]

Take \( y = X/2 \) and let \( X \) tend to \( \infty \). From the bounds on \( u_X \), the left hand side in the identity above remains bounded whereas the right hand side tends to infinity. We have obtained the desired contradiction. The last statement of Lemma 4.7 follows easily from the first statement of Proposition 4.1.

**Proposition 4.8.** If \((\alpha, \psi)\) satisfy Assumption 1 and if \( \sigma > 0 \), the function \( \mu_X, \) defined in Lemma 4.7, and satisfying the variational inequality (3.13) with \( \mu \in L^p(0, T; V_X) \), converges to \( \mu_X \) in \( L^p(0, T; \mathcal{V}_X) \), for all \( p, 1 \leq p < +\infty \). The sequence \( u_X, \) converges to \( u_X \) strongly in \( L^2(0, T; D_X) \) and in \( L^\infty(0, T; V_X) \).

**Proof.** See § 8.

### 4.3. The problem (VIP)

From Theorem 4.4, Proposition 4.6 and Lemma 4.7, we can pass to the limit as \( X \to \infty \):

**Theorem 4.9.** If \((\alpha, \psi)\) satisfy Assumption 1 and if \( \sigma > 0 \), there exists a unique solution of problem (VIP), i.e. a function \( u \in C^0([0, T]; K) \cap L^2(0, T; V^2) \), with \( \frac{\partial u}{\partial t} \in L^2((0, T) \times \mathbb{R}_+) \), such that \( u(t = 0) = u_o \),
\[
u(t, x) = 0, \quad \forall t \in [0, T], \ x \geq X_T, \tag{4.16}
\]

where \( X_T \) is defined in Lemma 4.7, and satisfying the variational inequality (3.13) for all \( \nu \in K \) with bounded support in \( x \). The function \( u \) coincides with \( u_X \) for \( X \geq X_T \). There exists a nondecreasing and lower semi-continuous function \( \gamma : (0, T] \to (S, X_T) \), such that \( \forall t \in (0, T), \ \{ x > 0 \ s.t. \ u(t, x) = u_o(x) \} = [\gamma(t), +\infty) \).

Calling
\[
\mu = \frac{\partial u}{\partial t} + Au + rx, \tag{4.17}
\]
we have a.e. \( 0 \leq \mu \leq rx1_{\{t=0\}} = rx1_{\{x \geq \gamma(t)\}} \).

**Proposition 4.10.** The function \( \mu \) defined in (4.17) is nondecreasing w.r.t. \( x \) (i.e. the distribution \( \frac{\partial \mu}{\partial t} \) is negative) and nonincreasing w.r.t. \( t \), (i.e. the distribution \( \frac{\partial \mu}{\partial t} \) is positive). For any \( X > X_T \), the total variation of \( \mu \) in \((0, T) \times (0, X)\) is bounded by \( rX(T + X) \).

**Proof.** Consider \( X > X_T \). The function \( \mu \) coincides with \( \mu_X \) on \((0, T) \times (0, X)\). The monotone character of \( \mu \) w.r.t. the two variables stems from (4.14) and from the fact that \( u \) is nonincreasing w.r.t. \( x \) and nondecreasing w.r.t. \( t \).

The same result can be proved by observing that \( \mu_X \) is the weak limit in \( L^2((0, T) \times
We can make the three observations:

1. The norm of $A$ as an operator from $V$ to $V'$ is bounded independently of $(\sigma, \alpha, \psi)$ in $F$.

2. The constants in (3.10)-(3.11) can be taken independent of $(\sigma, \alpha, \psi)$ in $F$.

3. With $\lambda$ in (3.10) independent of $(\sigma, \alpha, \psi)$ in $F$, the operator $A + \lambda I$ is one to one and continuous from $V^2$ onto $L^2(\mathbb{R}_+)$ and $(A + \lambda I)^{-1} : L^2(\mathbb{R}_+) \mapsto V^2$ is bounded with constants independent of $(\sigma, \alpha, \psi)$ in $F$.

By carefully inspecting the proofs of Theorems 4.2, 4.4 and 4.9, we see that

1. The quantities $\|u_{X,e}\|_{L^\infty(0,T;L^2(0,X))}$ and $\|u_{X,e}\|_{L^2(0,T;V_X)}$ are bounded independently of $(\sigma, \alpha, \psi)$ in $F$.

2. The quantities $\|\mathcal{E}_X(u_X)\|_{L^\infty(0,T;L^2(\mathbb{R}_+))}$ and $\|\mathcal{E}_X(u_X)\|_{L^2(0,T;V)}$ are bounded independently of $(\sigma, \alpha, \psi)$ in $F$.

3. The quantities $\|u\|_{L^\infty(0,T;L^2(\mathbb{R}_+))}$ and $\|u\|_{L^2(0,T;V)}$ are bounded independently of $(\sigma, \alpha, \psi)$ in $F$.

Proposition 4.12. The function $\gamma$ is bounded in $[0,T]$ by some constant $\hat{X}$ independent of $(\sigma, \alpha, \psi)$ in $F$. The quantities $\|u\|_{L^\infty(0,T;V)}$, $\|u\|_{L^2(0,T;V^2)}$ and $\|\frac{d}{dt}u\|_{L^2((0,T) \times \mathbb{R}_+)}$ are bounded independently of $(\sigma, \alpha, \psi)$ in $F$.

Proof. For a sequence $(u_n, \alpha_n, \psi_n)$ in $F$, let us call $u_n$ the corresponding solution of problem (VIP), and $\gamma_n$ the function such that $u_n(t, x) = 0 \Leftrightarrow x \geq \gamma_n(t)$. Assume that $\lim_{n \to \infty} \gamma_n(T/2) = +\infty$. Then, we can use the same arguments as in the proof of Lemma 4.7 and reach a contradiction. Therefore, $\gamma|_{[0,T/2]}$ is bounded independently of $(\sigma, \alpha, \psi)$ in $F$. Since (VIP) can always be solved in $(0, 2T) \times \mathbb{R}_+$ instead of $(0, T) \times \mathbb{R}_+$, and since for the solution $u$, $\|u\|_{L^2(0,2T;V)} + \|\frac{d}{dt}u\|_{L^2(0,2T;V^2)}$ is bounded independently of $(\sigma, \alpha, \psi)$ in $F$, we can use the same arguments and prove that $\gamma|_{[0,T]}$ is bounded independently of $(\sigma, \alpha, \psi)$ in $F$.

Therefore, it is possible to choose $\hat{X}$ such that, for any $(\sigma, \alpha, \psi) \in F$, $\gamma < \hat{X}$, and $u$ coincides with $\mathcal{E}_X(u_{\hat{X}})$ where $u_{\hat{X}}$ is the solution of (VIP$_{\hat{X}}$). For $x > \hat{X}$,

$$\mu(t, x) = rx - \int_{z > \log(x/\hat{X})} k(z)e^{z}u(xz^{-1})dz.$$
Thus, for $x$ large enough, such that, for example, $\log(x/\bar{X}) > 1$, 
\[
0 \leq rx - \mu(t, x) \leq S \int_{z > \log(x/\bar{X})} k(z)e^z dz \leq S \int_{z > \log(x/\bar{X})} \psi(z)e^{\psi z} dz \\
\leq S \bar{\psi} \int_{z > \log(x/\bar{X})} e^{\psi z} dz = \bar{\psi}S \frac{\bar{X}}{x}.
\]
(4.19)

Therefore, $\| \frac{\partial u}{\partial t} + Au \|_{L^2(0,T;L^2(\mathbb{R}))}$ is bounded by a constant independent of $(\sigma, \alpha, \psi)$. This implies that the quantities $\| u \|_{L^\infty(0,T;V)}$, $\| u \|_{L^2(0,T;V^2)}$ and $\| \frac{\partial u}{\partial t} \|_{L^2((0,T) \times \mathbb{R})}$ are bounded independently of $(\sigma, \alpha, \psi) \in \mathcal{F}$. \[\square\]

**Remark 9.** It may be possible to impose weaker conditions on $\psi$, and other choices of $\mathcal{F}$ could be made.

### 5. Sensitivity Analysis.
Here, we aim at understanding the sensitivity of the solution $u$ of (VIP) and of $\mu$ given by (4.17) to the variations of $(\sigma, \alpha, \psi) \in \mathcal{F}$. Let us introduce $\mathcal{B} = \{ f : z \mapsto f(z) \max(1, |z|^{b_1}, e^{2b_1z}) \in L^\infty(\mathbb{R}) \}$ endowed with the norm $\| f \|_\mathcal{B} = \| f(\cdot) \max(1, | \cdot |^{b_1}, e^{2b_1\cdot}) \|_{L^\infty(\mathbb{R})}$. For $(\sigma, \alpha, \psi) \in \mathcal{F}$, let $\mu(\sigma, \alpha, \psi)$ be the corresponding solution of (VIP). Accordingly, let $\mu(\sigma, \alpha, \psi)$ be given by (4.17) and $\gamma(\sigma, \alpha, \psi)$ be the function defining the free boundary.

**Proposition 5.1.** There exists $C > 0$, such that $(\sigma, \alpha, \psi) \in \mathcal{F}$, $(\tilde{\sigma}, \tilde{\alpha}, \tilde{\psi}) \in \mathcal{F}$,

\[
\| u - \tilde{u} \|_{L^2(0,T;V)} + \| u - \tilde{u} \|_{L^\infty(0,T;L^2(\mathbb{R}^+))} \leq C \left( |\sigma - \tilde{\sigma}| + |\alpha - \tilde{\alpha}| + \| \psi - \tilde{\psi} \|_{\mathcal{B}} \right),
\]
(5.1)

\[
\int_0^T \int_{\mathbb{R}} (\mu(\tilde{u} - u_\sigma) + \tilde{\mu}(u - u_\sigma)) \leq C \left( |\sigma - \tilde{\sigma}| + |\alpha - \tilde{\alpha}| + \| \psi - \tilde{\psi} \|_{\mathcal{B}} \right)^2,
\]
(5.2)

calling $u = u(\sigma, \alpha, \psi)$, $\mu = \mu(\sigma, \alpha, \psi)$, $\tilde{u} = u(\tilde{\sigma}, \tilde{\alpha}, \tilde{\psi})$, $\tilde{\mu} = \mu(\tilde{\sigma}, \tilde{\alpha}, \tilde{\psi})$.

**Proof.** We skip the proof since its arguments are well known. \[\square\]

**Proposition 5.2.** Consider $(\sigma, \alpha, \psi) \in \mathcal{F}$ and let $(\sigma_n, \alpha_n, \psi_n)_{n \in \mathbb{N}}$ be a sequence of coefficients in $\mathcal{F}$ such that $\lim_{n \to +\infty} (|\sigma - \sigma_n| + |\alpha - \alpha_n| + \| \psi - \psi_n \|_{\mathcal{B}}) = 0$. Calling $u = u(\sigma, \alpha, \psi)$, $u_n = u(\sigma_n, \alpha_n, \psi_n)$, $\mu = \mu(\sigma, \alpha, \psi)$ and $\mu_n = \mu(\sigma_n, \alpha_n, \psi_n)$,

\[
\lim_{n \to +\infty} \| u_n - u \|_{L^\infty((0,T) \times \mathbb{R}^+)} = 0, \quad \lim_{n \to +\infty} \| \mu_n - \mu \|_{L^p((0,T) \times \mathbb{R}^+)} = 0,
\]
(5.3)

for all $p$, $1 < p < +\infty$, and

\[
\| u_n - u \|_{L^\infty(0,T;V^1)} + \| u_n - u \|_{L^2(0,T;V^2)} + \| \frac{\partial (u_n - u)}{\partial t} \|_{L^2((0,T) \times \mathbb{R}^+)} \to 0.
\]
(5.4)

**Proof.** From the facts that

- $u(t,x) = u_n(t,x) = 0$ for $x > \bar{X}$ (where $\bar{X}$ is given in Proposition 4.12 and does not depend of $(\sigma, \alpha, \psi) \in \mathcal{F}$),
- $\forall n$, $S - x \leq u_n(t,x) \leq S$, and $S - x \leq u(t,x) \leq S$, which implies that $u - u_n$ is arbitrarily small as $x \to 0$ uniformly with respect to $n$,

it is enough to prove that $\forall \epsilon > 0$,

\[
\lim_{n \to +\infty} \| u_n - u \|_{L^\infty((0,T) \times (e,X))} = 0.
\]
(5.5)

From (5.1), we see that $\lim_{n \to \infty} \| u_n - u \|_{L^\infty(0,T;L^2(\mathbb{R}^+))} = 0$. On the other hand, we know that $\| \mu_n - rx \|_{L^\infty(0,T;V)}$ is bounded independently of $n$. These two observations imply (5.5), and the first part of (5.3) is proved.

Let us prove the second part of (5.3): from the fact that $\mu_n - rx$ is bounded $L^2((0,T) \times \mathbb{R}^+)$, one can extract a subsequence converging weakly in $L^2((0,T) \times \mathbb{R}^+)$. The limit is nothing else but $\mu - rx$, and the whole sequence $\mu_n - rx$ converges to
\[ \mu - rx \text{ weakly in } L^2((0, T) \times \mathbb{R}_+) \]. Thanks to (4.19) with \( \tilde{X} \) independent of \((\sigma, \alpha, \psi) \in \mathcal{F} \), it is enough to prove that \( \mu_n \) strongly converges to \( \mu \) in \( L^p((0, T) \times (0, X)) \), for any \( X > S \), and \( 1 < p < +\infty \). But, from Proposition 4.10, we know that the sequence \( (\mu_n) \) is bounded in \( BV((0, T) \times (0, X)) \) and in \( L^\infty((0, T) \times (0, X)) \), therefore compact in \( L^p((0, T) \times (0, X)) \), \( 1 \leq p < +\infty \). Therefore, a subsequence of \( (\mu_n) \) converges in \( L^p((0, T) \times (0, X)) \), and the limit is nothing but \( \mu \), from the observation above. The whole sequence \( (\mu_n) \), converges to \( \mu \) in \( L^p((0, T) \times (0, X)) \). We have proved that \( \mu_n - rx \) converges to \( \mu - rx \) in \( L^p((0, T) \times \mathbb{R}_+) \), \( 1 < p < +\infty \).

Finally, (5.4) follows from (5.3). \[ \square \]

6. The least square inverse problem. Let us introduce an Hilbert space \( H_\psi \) endowed with the norm \( \| \cdot \|_\psi \), relatively compact in \( B \).

Consider \( H_\psi \) a closed and convex subset of \( H_\psi \). We assume that \( H_\psi \) is contained in \( \{ \psi : \| \psi \|_B \leq \bar{\psi}; \psi \geq 0 \} \) and that a) the functions \( \psi \in H_\psi \) are continuous near 0, b) there exists two positive constants \( \bar{\psi} \) and \( \bar{\varepsilon} \) such that \( \psi(z) \geq \bar{\psi} \) for all \( z \) such that \( |z| \leq \bar{\varepsilon} \), c) there exist two constant \( \bar{\eta} > 0 \) and \( C \geq 0 \) such that for all \( \psi \in H_\psi \), \( \psi(z) e^{\bar{\eta}/2z} - \psi(0) e^{-3/2z} = \omega(z) \), with \( |\omega(z)| \leq C|z| e^{-\bar{\eta}|z|} \), for all \( z \in \mathbb{R} \). This choice of \( H_\psi \) will allow us to use the results stated in Lemma 3.6.

Finally, consider the set \( \mathcal{H} = [\sigma, \bar{\sigma}] \times [-1/2, 1 - \alpha] \times H_\psi \). Let \( J_R \) be a convex, coercive and \( C^1 \) function defined on \([\sigma, \bar{\sigma}] \times [-1/2, 1 - \alpha] \times H_\psi \). It is well known that \( J_R \) is also weakly lower semicontinuous. The functional \( J_R \) may depend on suitable prior parameters \( \sigma_0, \alpha_0 \) and \( \psi_0 \). It is the analog of the function \( \rho J_2 \) discussed in § 1.

6.1. Toward the calibration problem.

6.1.1. Orientation. For calibrating the Lévy process, one observes the spot price \( S \) and the prices \( (\bar{p}_i)_{i \in I} \) of a family of American put options with maturities/strikes given by \( (T_i, x_i) \), and we call \( \bar{u}_i = \bar{p}_i - x_i + S \), \( i \in I \). The parameters of the Lévy process, i.e. the volatility \( \sigma \), the exponent \( \alpha \) and the function \( \psi \) will be found as solutions of a least square problem, where the functional to be minimized is the sum of a suitable Tychonoff regularization functional and of

\[ J(u) = \sum_{i \in I} \omega_i (u(T_i, x_i) - \bar{u}_i)^2, \]

where \( \omega_i \) are positive weights, and \( u = u(\sigma, \alpha, \psi) \) is a solution of (VIP).

We aim at finding some necessary optimality conditions satisfied by the solutions of the least square problem. The main difficulty comes from the fact that the derivability of the functional \( J(u) \) with respect to the parameter \((\sigma, \alpha, \psi) \) is not guaranteed. To obtain some necessary optimality conditions, we shall consider first a least square problem where \( u \) is the solution of the penalized problem (6.6) rather than (VIP), obtain the necessary optimality conditions for this new problem, then have the penalty parameter \( \epsilon \) tend to 0 and pass to the limit in the optimality conditions. Such a program has already been applied in [2] for calibrating the local volatility with American options, see also [4, 5] for a related numerical method and results. The idea originally comes from Hintermüller [22] and Ito and Kunisch [23], who applied a similar program for elliptic variational inequalities. Let us also mention Mignot and Puel [32] who applied a nice method for finding the optimality conditions of a special control problem with a parabolic variational inequality.

In order to simplify the notations, we are going to consider first a toy problem where only one price is observed. Of course, observing one price only is not enough. However, finding the optimality conditions for this simplified calibration problem presents the same difficulties as for the original one.

6.1.2. The least square problem and its penalized version. A first step towards the calibration problem is to consider the functional

\[ J : C^0([0, T] \times \mathbb{R}_+) \rightarrow \mathbb{R}, \quad J(u) = (u(T, x_{ob}) - \bar{u})^2, \quad (6.1) \]
where \( x_{ob} \) and \( \bar{u} \) are positive numbers. We fix \( \bar{X} \) (independent of \((\sigma, \alpha, \psi) \in \mathcal{H}\)) as in Proposition 4.12 and assume that \( x_{ob} < \bar{X} \). Consider the least square problem:

\[
\text{Minimize } J(u) + J_R(\sigma, \alpha, \psi) \quad (\sigma, \alpha, \psi) \in \mathcal{H}, u = u(\sigma, \alpha, \psi) \text{ satisfies (VIP)}.
\]  

(6.2)

Fixing \( X \geq \bar{X} \), we know that \( u_{|[0,T]\times [0,X]} = u_X \) where \( u_X \) is the solution of (VIP\(_X\)). Therefore, (6.2) is equivalent to the least square problem:

\[
\text{Minimize } J(u) + J_R(\sigma, \alpha, \psi) \quad (\sigma, \alpha, \psi) \in \mathcal{H}, u \text{ satisfies (VIP\(_X\))}.
\]  

(6.3)

We will also consider the least square problem related to the penalized problem

\[
\text{Minimize } J(u_\epsilon) + J_R(\sigma, \alpha, \psi) \quad (\sigma, \alpha, \psi) \in \mathcal{H}, u_\epsilon \text{ satisfies (4.6)}.
\]  

(6.4)

**Lemma 6.1.** Let \((\epsilon_n)_n\) be a sequence of penalty parameters such that \( \epsilon_n \to 0 \) as \( n \to \infty \), and let \((\sigma_{\epsilon_n}^*, \alpha_{\epsilon_n}^*, \psi_{\epsilon_n}^*)\) be a solution of problem (6.4). Consider a subsequence such that \((\sigma_{\epsilon_n}^*, \alpha_{\epsilon_n}^*, \psi_{\epsilon_n}^*)\) converges to \((\sigma^*, \alpha^*, \psi^*)\) in \( \mathcal{F} \), \( \psi_{\epsilon_n}^* \) weakly converges to \( \psi^* \) in \( H_0 \), and \( u_{\epsilon_n}^* \to u^* \) weakly in \( L^2(0,T;V_X) \), where \( V_X \) is defined in (4.1). Then \((\sigma^*, \alpha^*, \psi^*), u^*\) is a solution of (6.3). We have that

- \( u_{\epsilon_n}^* \) converges to \( u^* \) uniformly in \( [0,T] \times [0,X] \), and in \( L^2(0,T;V_X) \).
- \( 1_{\{x > 3\}} x^2 \chi u_{\epsilon_n}^* \) converges to \( \mu^* \) strongly in \( L^2((0,T) \times (0,X)) \).
- For all smooth functions \( \chi \) with compact support contained in \([0,X]\), \( \chi \mathcal{E}_X(u_{\epsilon_n}^*) \) converges to \( \chi \mathcal{E}_X(u^*) \) strongly in \( L^2(0,T;V^2) \) and in \( L^\infty(0,T;V) \).

**Proof.** For brevity, the proof is outlined only. We skip the proof that \( u^* \) satisfies (VIP\(_X\)) with \((\sigma, \alpha, \psi) = (\sigma^*, \alpha^*, \psi^*)\) and the proofs of the first two points above, since they are in the same spirit as the proofs of Theorem 4.4 and Proposition 4.8. The third point above is proved by writing the boundary value problems satisfied by \( y_n = \chi \mathcal{E}_X(u_{\epsilon_n}^*) \) and \( y = \chi \mathcal{E}_X(u^*) \), with the PID equations

\[
\frac{\partial y_n}{\partial t} + A_n y_n = f_n, \quad \frac{\partial y}{\partial t} + Ay = f,
\]

where \( A \) (resp. \( A_n \)) is given by (3.9) and (2.12) with \((\sigma, \alpha, \psi) = (\sigma^*, \alpha^*, \psi^*)\) (resp. \((\sigma_{\epsilon_n}^*, \alpha_{\epsilon_n}^*, \psi_{\epsilon_n}^*)\)) and where the right hand side \( f \) (resp. \( f_n \)) can be written in terms of \( \chi, u^* \) and \( \mu^* \) (resp. \( \chi \) and \( u_{\epsilon_n}^* \)). By using the first two points above and the same arguments as in the proofs of Propositions 5.1 and 5.2, it can be proved that \( f_n \) converges to \( f \) in \( L^2((0,T) \times \mathbb{R}_+) \), and that \( y_n \) converges to \( y \) in \( L^2(0,T;V^2) \) and in \( L^\infty(0,T;V) \).

As a consequence of the first point above, \( J(u_{\epsilon_n}^*) \to J(u^*) \). Moreover, from the assumptions on \( J_R \), \( J_R(\sigma^*, \alpha^*, \psi^*) \leq \liminf_{n \to \infty} J_R(\sigma_{\epsilon_n}^*, \alpha_{\epsilon_n}^*, \psi_{\epsilon_n}^*) \).

Since \((\sigma_{\epsilon_n}^*, \alpha_{\epsilon_n}^*, \psi_{\epsilon_n}^*)\) is a solution of (6.4),

\[
J(u_{\epsilon_n}^*) + J_R(\sigma_{\epsilon_n}^*, \alpha_{\epsilon_n}^*, \psi_{\epsilon_n}^*) \leq J(u_{\epsilon_n}(\sigma, \alpha, \psi)) + J_R(\sigma, \alpha, \psi) \quad \forall (\sigma, \alpha, \psi) \in \mathcal{H},
\]

where \( u_{\epsilon_n}(\sigma, \alpha, \psi) \) is the solution of (4.6) with \( \epsilon = \epsilon_n \). This implies that

\[
J(u^*) + J_R(\sigma^*, \alpha^*, \psi^*) \leq J(u(\sigma, \alpha, \psi)) + J_R(\sigma, \alpha, \psi) \quad \forall (\sigma, \alpha, \psi) \in \mathcal{H},
\]

where \( u(\sigma, \alpha, \psi) \) satisfies (VIP\(_X\)) and \((\sigma^*, \alpha^*, \psi^*), u^* \) is a solution of (6.3). \( \square \)

**Remark 10.** Let \((\sigma_{\epsilon_n}^*, \alpha_{\epsilon_n}^*, \psi_{\epsilon_n}^*)\), \( u_{\epsilon_n}^* \) be a subsequence converging to \((\sigma^*, \alpha^*, \psi^*), u^* \) as in Lemma 6.1. It is clear from the continuity of \( u^* \) and from the uniform convergence of \( u_{\epsilon_n}^* \) that if \( u^*(T,x_{ob}) > u_\epsilon(x_{ob}) \), then there exists a constant \( a > 0 \) and an integer \( N \) such that for \( n > N \), \( u_{\epsilon_n}^*(t,x) > u_\epsilon(x) + \epsilon_n \) for all \((t,x)\) with \( |x - x_{ob}| < a \) and \( t > T - a \).
6.1.3. First order necessary optimality conditions for (6.4). We take \((\sigma_u^*, \alpha_u^*, \psi_u^*, u_n^*, \psi_n^*)\) and \((\sigma^*, \alpha^*, \psi^*), u^*\) as in Lemma 6.1. We assume that \(u^*(T, x_{ob}) > u_o(x_{ob})\) and we take \(N\) and \(a\) as in Remark 10. For \(n > N\), we wish to find necessary optimality conditions for the solution \((\sigma_{n}^*, \alpha_{n}^*, \psi_{n}^*), u_{n}^*\) of (6.4). In order to simplify the notations, we drop the index \(n\): below, \(\epsilon\) means \(\epsilon_n\).

We shall need to solve an adjoint problem. Since the cost functional involves pointwise values of \(u\), the adjoint problem will have a singular data. In that context, the notion of very weak solution of boundary value problems will be relevant: we introduce the space \(Z_\epsilon = \{ v \in \tilde{Z}_\epsilon; v(t = 0) = 0 \}\), where

\[
\tilde{Z}_\epsilon = \left\{ v \in L^2(0, T; V_N); \frac{\partial v}{\partial t} + A_{\epsilon, X} v - r x_1(x_{>\epsilon}) V'(u_0^*) v \in L^2((0, T) \times (0, X)) \right\}
\]

and \(A_{\epsilon, X}\) is the operator defined by (4.2), with \((\sigma^*, \alpha^*, \psi^*) = (\sigma_u^*, \alpha_u^*, \psi_u^*)\). The space \(Z_\epsilon\) endowed with the graph norm is a Banach space.

**Lemma 6.2.** Assume that \(u^*(T, x_{ob}) > u_o(x_{ob})\) and take \(N\) and \(a\) as in Remark 10. There exists a unique \(p_\epsilon^* \in L^2((0, T) \times (0, X))\) such that, for all \(v \in Z_\epsilon\),

\[
\int_0^T \int_0^X \left( \frac{\partial p_\epsilon^*}{\partial t} + A_{\epsilon, X} p_\epsilon^* - r x_1(x_{>\epsilon}) V'(u_0^*) p_\epsilon^* \right) v \, dX \, dt = 0,
\]

and \(\|p_\epsilon^*\|_{L^2((0, T) \times (0, X))}\) is bounded by a constant independent of \(\epsilon\) in the subsequence.

**Proof.** See §8. \(\square\)

**Remark 11.** Problem (6.5) is a very weak formulation of:

\[
\begin{align*}
\frac{\partial p_\epsilon^*}{\partial t} - A_{\epsilon, X}^T p_\epsilon^* + r x_1(x_{>\epsilon}) V'(u_0^*) p_\epsilon^* &= 0, \\
p_\epsilon^*(t, x) &= 0, t \in (0, T), \\
p_\epsilon^*(T) &= -2(u_0^*(T, x_{ob}) - \bar{u})\delta_{x_{ob}},
\end{align*}
\]

where \(A_{\epsilon, X}^T(x) = -\frac{\sigma_u^*}{2} \frac{\partial^2}{\partial x^2} (x^2) + B_{\epsilon, X}^T v(x) - \frac{\partial}{\partial x} (r x v)\) with

\[
B_{\epsilon, X}^T v(x) = \int_X k_\epsilon^*(z) \left( z(x^z - 1) \frac{\partial v}{\partial x}(x) - 1 - \log(z) \frac{d}{dz} v(x + \frac{1}{z} - 1) + (2e^z - 1) v(x) \right) dz,
\]

and \(k_\epsilon^*(z) = |z|^{-2(\alpha_u^* + 1)} \psi_u^*(z)\).

**Remark 12.** Similarly, \(\frac{\partial (\phi p_\epsilon^*)}{\partial t} - A_{\epsilon, X}^T (\phi p_\epsilon^*) + r x_1(x_{>\epsilon}) V'(u_0^*) (\phi p_\epsilon^*) \|_{L^2((0, T) \times (0, X))}\) is bounded independently of \(\epsilon\).

From Lemma 6.2, we see that \(\int_0^T \int_X \langle x^2 \frac{\partial^2 u_0^*}{\partial x^2}, \phi p_\epsilon^* \rangle\) is well defined, where \(\langle, \rangle\) is the duality pairing between \((V_N)'\) and \(V_N\). On the other hand, \(\int_0^T \int_X \langle (1 - \phi)x^2 \frac{\partial^2 u_0^*}{\partial x^2}, p_\epsilon^* \rangle\) is well defined since both \((1 - \phi)x^2 \frac{\partial^2 u_0^*}{\partial x^2}\) and \(p_\epsilon^*\) are square integrable. Moreover, the sum \(\int_0^T \int_X \langle x^2 \frac{\partial^2 u_0^*}{\partial x^2}, \phi p_\epsilon^* \rangle + \int_0^T \int_X \langle (1 - \phi)x^2 \frac{\partial^2 u_0^*}{\partial x^2}, p_\epsilon^* \rangle\) does not depend on the choice of \(\phi\). Therefore, we call \(G(\epsilon)(u_\epsilon^*, p_\epsilon^*)\) the quantity

\[
G(\epsilon)(u_\epsilon^*, p_\epsilon^*) = \int_0^T \int_X \left( x^2 \frac{\partial^2 u_0^*}{\partial x^2}, \phi p_\epsilon^* \right) + \int_0^T \int_X \left( (1 - \phi)x^2 \frac{\partial^2 u_0^*}{\partial x^2}, p_\epsilon^* \right)\]

Let us introduce the operator \(B_{\epsilon, X}^{(\phi)}\):

\[
B_{\epsilon, X}^{(\phi)} v(x) = -\int_R k_\epsilon^*(z) \log(|z|) \left( (x^z - 1) \frac{\partial v}{\partial x}(x) + x^z (1 - \log(z)) \frac{d}{dz} v(x + \frac{1}{z}) - v(x) \right) dz,
\]
where $k^*_i(z) = |z|^{-2\alpha^*_i-1}\psi^*_i(z)$. From Lemma 6.2, the quantity $\int_0^T \langle B^{(\sigma)}_{\epsilon,X} u^*_\epsilon, \phi^*_\epsilon \rangle$ is well defined, where $\langle , \rangle$ is the duality pairing between $(V_X)'$ and $V_X$. On the other hand, the quantity $\int_0^T \int_X ( (1-\phi) B^{(\sigma)}_{\epsilon,X} u^*_\epsilon ) p^*_\epsilon$ is well defined since both $p^*_\epsilon$ and $((1-\phi) B^{(\sigma)}_{\epsilon,X} u^*_\epsilon)$ are square integrable. Moreover, the sum $\int_0^T \int_X ( (1-\phi) B^{(\sigma)}_{\epsilon,X} u^*_\epsilon ) p^*_\epsilon$ does not depend on $\phi$. Therefore, we denote $G^{(\sigma)}_{\epsilon}(u^*_\epsilon, p^*_\epsilon)$ the quantity

$$G^{(\sigma)}_{\epsilon}(u^*_\epsilon, p^*_\epsilon) = \int_0^T \langle B^{(\sigma)}_{\epsilon,X} u^*_\epsilon, \phi^*_\epsilon \rangle + \int_0^T \int_X ( (1-\phi) B^{(\sigma)}_{\epsilon,X} u^*_\epsilon ) p^*_\epsilon. \quad(6.9)$$

Similarly, for $\kappa \in H^\phi$, we introduce the operator $B^{(\psi,\kappa)}_{\epsilon,X}:

$$B^{(\psi,\kappa)}_{\epsilon,X} v(x) = \int_R \frac{\kappa(z)}{|z|^{1+2\alpha^*_i}} \left( x(e^z-1) \frac{\partial v}{\partial x}(x) + e^z(1_{(z> -\log(x))}v(xe^{-z}) - v(x)) \right) dz,$$

and the quantity

$$\langle G^{(\psi)}_{\epsilon}(u^*_\epsilon, p^*_\epsilon), \kappa \rangle = \int_0^T \langle B^{(\psi,\kappa)}_{\epsilon,X} u^*_\epsilon, \phi^*_\epsilon \rangle + \int_0^T \int_X ( (1-\phi) B^{(\psi,\kappa)}_{\epsilon,X} u^*_\epsilon ) p^*_\epsilon, \quad(6.10)$$

which does not depend on $\phi$. We are now ready to give necessary optimality for the least square problem (6.4):

**Proposition 6.3.** The optimality conditions for problem (6.4) are: for all $(\sigma, \alpha, \psi) \in H$,

$$(\sigma - \sigma^*_\epsilon) \left( D_{\sigma} J_R(\sigma^*_\epsilon, \alpha^*_\epsilon, \psi^*_\epsilon) + \sigma^*_\epsilon G^{(\sigma)}(u^*_\epsilon, p^*_\epsilon) \right) \geq 0. \quad(6.11)$$

$$(\alpha - \alpha^*_\epsilon) \left( D_{\alpha} J_R(\sigma^*_\epsilon, \alpha^*_\epsilon, \psi^*_\epsilon) + 2G^{(\sigma)}_{\epsilon}(u^*_\epsilon, p^*_\epsilon) \right) \geq 0. \quad(6.12)$$

$$(D_{\psi} J_R(\sigma^*_\epsilon, \alpha^*_\epsilon, \psi^*_\epsilon), \psi - \psi^*_\epsilon) + \left( G^{(\psi)}_{\epsilon}(u^*_\epsilon, p^*_\epsilon), \psi - \psi^*_\epsilon \right) \geq 0. \quad(6.13)$$

**Proof.** The proof is quite standard. It is omitted for brevity. $\Box$

6.1.4. First order necessary optimality conditions for (6.3). In order to obtain optimality conditions for (6.3), we wish to pass to the limit in the optimality conditions for (6.4). Let $\epsilon_n$ be sequence of penalty parameters converging to zero, and let $(\sigma^*_{\epsilon_n}, \alpha^*_{\epsilon_n}, \psi^*_{\epsilon_n}, u^*_{\epsilon_n})$ be a sequence of solutions to (6.4) converging to $(\sigma^*, \alpha^*, \psi^*, u^*)$ as in Lemma 6.1. Assume that there exists a positive number $a$ such that $u^*_{\epsilon_n}(t,x) > u_a(x)+\epsilon_n$ for all $(t,x)$ with $|x - x_{ob}| \leq a$ and $T - t \leq a$. Let $p^*_{\epsilon_n}$ be the adjoint state defined by Lemma 6.2. There exists a subsequence denoted $n_k$ such that $p^*_{\epsilon_n_k}$ weakly converges to $p^*$ in $L^2((0,T) \times (0,X))$ and $\phi p^*_{\epsilon_n_k}$ weakly converges to $\phi p^*$ in $L^2(0,T;V_X)$, where $\phi$ is given in Lemma 6.2.

We call $\bar{Z}$ and $Z$ the spaces

$$\bar{Z} = \left\{ v \in L^2(0,T;V_X); \frac{\partial v}{\partial t} + A_X v \in L^2((0,T) \times (0,X)) \right\},$$

$$Z = \left\{ v \in \bar{Z}; v(t=0) = 0 \right\}, \quad(6.14)$$

where $A_X$ is the operator given by (4.2), (3.9) and (2.12), with the parameter $(\sigma^*, \alpha^*, \psi^*)$. These spaces, endowed with the graph norm, are Banach spaces.

**Proposition 6.4.** There exists a Radon measure $\xi^*$ such that for all $v \in Z$,

$$\int_0^T \int_0^X \left( \frac{\partial v}{\partial t} + A_X v \right) p^* + <\xi^*, v > = 2(u^*(T,x_{ob}) - \bar{u})v((T,x_{ob})). \quad(6.15)$$
The function $p^*$ satisfies
\[
\frac{\partial p^*}{\partial t} - A^T_X p^* - \xi^* = 0 \tag{6.16}
\]
in the sense of distributions. Furthermore, with $u^*$, $\mu^*$ defined as in Lemma 6.1,
\[
\mu^*|p^*| = 0, \tag{6.17}
\]
\[
|u^*|\xi^* = 0. \tag{6.18}
\]

Proof. For simplicity, we drop the index $n$ in $\epsilon_n$. In what follows, $\epsilon$ means $\epsilon_n$. For a positive parameter $\delta$, we introduce the nondecreasing function $\rho_\delta : \mathbb{R} \to \mathbb{R}$:
\[
\rho_\delta(p) = -1 \quad \text{for} \quad p \leq -\delta, \quad \rho_\delta(p) = p/\delta \quad \text{for} \quad -\delta \leq p \leq \delta, \quad \rho_\delta(p) = 1 \quad \text{for} \quad p \geq \delta.
\]
and the nonnegative function $R_\delta(p) = \int_0^p \rho_\delta(q) dq$.
In what follows, $\delta$ will be the generic term of a decreasing sequence of positive parameters which converges to 0.
For $\phi$ introduced in Lemma 6.2, we use Remark 12: there exists a function $g_\epsilon \in L^2((0, T) \times (0, X))$ with a norm bounded independently of $\epsilon$ such that $\phi p^*_\epsilon$ is the weak solution to
\[
\frac{\partial (\phi p^*_\epsilon)}{\partial t} - A^T_X (\phi p^*_\epsilon) + r x 1_{\{x > s\}} V'(u^*_\epsilon) (\phi p^*_\epsilon) = g_\epsilon
\]
with the Cauchy condition $(\phi p^*_\epsilon)(T, \cdot) = 0$ and the boundary condition $(\phi p^*_\epsilon)(\cdot, X) = 0$. Therefore, $\|\phi p^*_\epsilon\|_{L^2(0, T; V_X)}$ is bounded uniformly in $\epsilon$. Moreover, from the properties of $\phi$, see Lemma 6.2, we have $V'(u^*_\epsilon) (\phi p^*_\epsilon) = V'_\epsilon (u^*_\epsilon) p^*_\epsilon$.
Multiplying the last equation by $\rho_\delta(\phi p^*_\epsilon)$, we obtain that there exists a constant $C$ independent of $\delta$ and $\epsilon$ such that
\[
\int_0^X R_\delta(\phi p^*_\epsilon)(0, x) dx + \int_0^T \int_0^X \frac{(\sigma^*)^2 x^2}{2} \rho_\delta(\phi p^*_\epsilon) \left( \frac{\partial (\phi p^*_\epsilon)}{\partial x} \right)^2 dx - r \int_0^T \int_0^X x V'_\epsilon(u^*_\epsilon) p^*_\epsilon \rho_\delta(\phi p^*_\epsilon) + \int_0^T \langle (B^T_{\epsilon, X} (\phi p^*_\epsilon)), \rho_\delta(\phi p^*_\epsilon) \rangle \leq C. \tag{6.19}
\]
Let us focus on the last term in the sum above: we can write it as
\[
\int_0^T \int_{\mathbb{R}_+} \langle (B_{\epsilon} - B^T_{\epsilon}) (\partial_x (\phi p^*_\epsilon)), \rho_\delta(\phi p^*_\epsilon) \rangle \geq \frac{1}{2} \int_0^T \langle (B_{\epsilon} + B^T_{\epsilon}) (\partial_x (\phi p^*_\epsilon)), \rho_\delta(\phi p^*_\epsilon) \rangle - \frac{1}{2} \int_0^T \langle (B_{\epsilon} - B^T_{\epsilon}) (\partial_x (\phi p^*_\epsilon)), \rho_\delta(\phi p^*_\epsilon) \rangle. \tag{6.20}
\]
From Lemma 3.6 and the choice of $\mathcal{H}_\psi$, there exists a constant $C$ independent of $(\sigma, \alpha, \psi) \in \mathcal{H}$ and of $\delta$ such that
\[
\int_0^T \langle (B_{\epsilon} - B^T_{\epsilon}) (\partial_x (\phi p^*_\epsilon)), \mathcal{E}_X(\rho_\delta(\phi p^*_\epsilon)) \rangle \leq \|\phi p^*_\epsilon\|_{L^2((0, T); V_X)} \|\rho_\delta(\phi p^*_\epsilon)\|_{L^2((0, T) \times (0, X))} \leq C. \tag{6.21}
\]
On the other hand, from Lemma 3.5,
\[
\int_0^T \langle (B_{\epsilon} + B^T_{\epsilon}) (\partial_x (\phi p^*_\epsilon)), \mathcal{E}_X(\rho_\delta(\phi p^*_\epsilon)) \rangle + \int_0^T \int_0^X k_\epsilon(z) e^{\psi} \left( (\phi p^*_\epsilon)(x) - (\phi p^*_\epsilon)(xe^{-\psi}) \right) (\rho_\delta(\phi p^*_\epsilon)(x) - \rho_\delta(\phi p^*_\epsilon)(xe^{-\psi})) \leq \|\phi p^*_\epsilon\|_{L^2((0, T) \times (0, X))} \|\rho_\delta(\phi p^*_\epsilon)\|_{L^2((0, T) \times (0, X))} \leq C. \tag{6.22}
\]
From (6.19), (6.20), (6.21) and (6.22), we see that

\[
\int_0^X R_\delta(\phi p^*_\epsilon)(0, x)\,dx
\]

\[+
\int_0^T \int_0^X \int_{\mathbb{R}} k_\epsilon(z)e^z \left( (\phi p^*_\epsilon)(x) - (\phi p^*_\epsilon)(xe^{-z}) \right) (\rho_\delta(\phi p^*_\epsilon)(x) - \rho_\delta(\phi p^*_\epsilon)(xe^{-z}))
\]

\[+
\int_0^T \int_0^X \left( \frac{(\sigma^*_\epsilon)^2x^2}{2} \rho_\delta(\phi p^*_\epsilon) \left( \frac{\partial(\phi p^*_\epsilon)}{\partial x} \right)^2 \right) - r \int_0^T \int_S xV'_\epsilon(u^*_\epsilon)p^*_\epsilon \rho_\delta(p^*_\epsilon) \leq C.
\]

Since \( p \mapsto p\rho_\delta(p) \) is a nonnegative function and since \( \rho_\delta \) is nondecreasing, all the terms in the sum above are nonnegative. Therefore, for a constant \( C \) independent of the parameters, \(-r \int_0^T \int_S X xV'_\epsilon(u^*_\epsilon)p^*_\epsilon \rho_\delta(p^*_\epsilon) \leq C.
\]

On the other hand we know that \(-xV'_\epsilon(u^*_\epsilon)p^*_\epsilon \rho_\delta(p^*_\epsilon) \) defines an increasing (as \( \delta \) decreases) sequence of nonnegative functions, which converges almost everywhere to \( xV'_\epsilon(u^*_\epsilon)p^*_\epsilon \) as \( \delta \to 0 \). Thus, Beppo Levi’s theorem tells us that

\[-r \int_0^T \int_S X xV'_\epsilon(u^*_\epsilon)p^*_\epsilon \rho_\delta(p^*_\epsilon) \] tends to \( r \int_0^T \int_S X xV'_\epsilon(u^*_\epsilon)p^*_\epsilon \) as \( \delta \to 0 \). Therefore, for a positive constant \( C \),

\[r \int_0^T \int_S X x|V'_\epsilon(u^*_\epsilon)p^*_\epsilon| \leq C. \quad (6.23)
\]

It is thus possible to extract a subsequence \( \epsilon_n_k \), such that \( p^*_\epsilon_n_k \to p^* \) weakly in \( L^2((0, T) \times (0, X)) \), \( \phi p^*_\epsilon_n_k \to \phi^* \) weakly in \( L^2(0, T; \mathcal{V}_X) \), and \(-r x1_{\{x>S\}} V'_\epsilon(u^*_\epsilon) p^*_\epsilon \) converges to \( \xi^* \) weakly* in \( (L^\infty((0, T) \times (0, X)))^* \). In order to simplify the notations, we omit the indexes \( n_k \); now, \( \epsilon \) means \( \epsilon_n_k \).

From this, (6.15) is obtained as well by passing to the limit in (6.5), and (6.16) is satisfied in the sense of distributions.

For proving (6.17), we use the convexity of \( \mathcal{V}_\epsilon \), (still dropping the index \( n_k \) in \( \epsilon_n_k \)): since \( \mathcal{V}_\epsilon(\epsilon) = 0 \), we have that for all \( u \in [0, \epsilon] \), \( \mathcal{V}_\epsilon(u) \leq -\mathcal{V}'_\epsilon(u)(\epsilon - u) \leq -\epsilon\mathcal{V}'_\epsilon(u) \).

This implies that \( \mathcal{V}_\epsilon(u^*_\epsilon) \leq -\epsilon\mathcal{V}'_\epsilon(u^*_\epsilon) \) because we also know that \( \mathcal{V}_\epsilon(u^*_\epsilon) = 0 \) if \( u^*_\epsilon \geq \epsilon \). Thus, calling \( \mu^*_\epsilon = -r x1_{\{x>S\}} \mathcal{V}'_\epsilon(u^*_\epsilon) \), (6.23) implies that

\[
0 \leq \int_0^T \int_0^X \mu^*_\epsilon |p^*_\epsilon| - \epsilon r \int_0^T \int_S X xV'_\epsilon(u^*_\epsilon)|p^*_\epsilon| \to 0. \quad (6.24)
\]

But we also know that \( p^*_\epsilon \to p^* \) weakly in \( L^2((0, T) \times (0, X)) \) and that \( \mu^*_\epsilon \to \mu^* \) strongly in \( L^2((0, T) \times (0, X)) \) from Lemma 6.1. Hence, \( \int_0^T \int_0^X \mu^*_\epsilon |p^*_\epsilon| \to \int_0^T \int_0^X \mu^* |p^*| \), and (6.17) is proved.

Let us call \( \xi^*_\epsilon = -r x1_{\{x>S\}} \mathcal{V}'_\epsilon(u^*_\epsilon) p^*_\epsilon = -r x1_{\{x>S\}} V'_\epsilon(u^*_\epsilon) \phi p^*_\epsilon \); for \( \chi \) a continuous function in \( [0, T] \times [0, X] \), we have

\[
\int_0^T \int_0^X |\xi^*_\epsilon||\chi u^*_\epsilon| \leq r \left( \int_0^T \int_S X |xV'_\epsilon(u^*_\epsilon)||\phi p^*_\epsilon|^2 \right)^{\frac{1}{2}} \left( \int_0^T \int_S X |xV'_\epsilon(u^*_\epsilon)||\chi u^*_\epsilon|^2 \right)^{\frac{1}{2}}
\]

from the Cauchy-Schwarz inequality. But it can be checked that \( |V'_\epsilon(u^*_\epsilon)||u^*_\epsilon\chi|^2 \leq C\epsilon \), which yields \( \int_0^T \int_S X |xV'_\epsilon(u^*_\epsilon)||u^*_\epsilon\chi|^2 \leq C\epsilon \). On the other hand, it is easy to check that \( \int_0^T \int_0^X X |xV'_\epsilon(u^*_\epsilon)||\phi p^*_\epsilon|^2 \leq C \). Therefore \( \int_0^T \int_{\mathbb{R}} \xi^*_\epsilon |u^*_\epsilon| \chi \to 0 \) as \( \epsilon \to 0 \). We know that \( \xi^*_\epsilon \to \xi^* \) weakly in \( (L^\infty)^* \) and that \( |u^*_\epsilon| \chi \to |u^*| \chi \) in \( C^0([0, T] \times [0, X]) \) from Lemma 6.1. We can pass to the limit as \( \epsilon \to 0 \) and (6.18) is proved. □
Proceeding as in (6.7), (6.9), (6.10), we introduce the quantities

\[
\mathcal{G}^{(\sigma)}(u^*, p^*) = \int_0^T \left\langle x^2 \frac{\partial^2 u^*}{\partial x^2}, \phi p^* \right\rangle + \int_0^T \int_X \left( (1 - \phi) x^2 \frac{\partial^2 u^*}{\partial x^2} \right) p^*,
\]

(6.25)

\[
\mathcal{G}^{(\alpha)}(u^*, p^*) = \int_0^T \left\langle B_X^{(\alpha)} u^*, \phi p^* \right\rangle + \int_0^T \int_X \left( (1 - \phi) B_X^{(\alpha)} u^* \right) p^*,
\]

(6.26)

\[
\left\langle \mathcal{G}(\psi)(u^*, p^*), \kappa \right\rangle = \int_0^T \left\langle B_X^{(\psi, \kappa)} u^*, \phi p^* \right\rangle + \int_0^T \int_X \left( (1 - \phi) B_X^{(\psi, \kappa)} u^* \right) p^*(6.27)
\]

where \(\phi\) is chosen as in Lemma 6.2, and where

\[
B_X^{(\alpha)}(v) = - \int_{\mathbb{R}} k^*(z) \log(|z|) \left( x(e^z - 1) \frac{\partial v}{\partial x}(x) + e^z(1 + \log(z)) v(xe^{-z}) - v(x) \right),
\]

\[
B_X^{(\psi, \kappa)}(v) = \int_{\mathbb{R}} \frac{k^*(z)}{|z|^{1+2\alpha}} \left( x(e^z - 1) \frac{\partial v}{\partial x}(x) + e^z(1 + \log(z)) v(xe^{-z}) - v(x) \right) dz.
\]

One can check exactly as above that \(\mathcal{G}^{(\sigma)}(u^*, p^*), \mathcal{G}^{(\alpha)}(u^*, p^*)\) and \(\left\langle \mathcal{G}(\psi)(u^*, p^*), \kappa \right\rangle\) are well defined and do not depend of the particular choice of \(\phi\). We are now ready to give necessary optimality for the least square problem (6.4):

**PROPOSITION 6.5.** Let \((\sigma^*, \alpha^*, \psi^*, u^*)\) be a solution to problem (6.3) obtained in Lemma 6.1. Assume that \(u^*(T, x_{ob}) > u_0(x_{ob})\) and take \(a\) as in Remark 10. There exist \(p^* \in L^2((0, T) \times (0, X))\) and a Radon measure \(\xi^*\) satisfying (6.15), (6.17), (6.18), and such that, for all \((\sigma, \alpha, \psi) \in \mathcal{H},\)

\[
(\sigma - \sigma^*) \left( D_{\sigma} J_R(\sigma^*, \alpha^*, \psi^*) + \sigma^* \mathcal{G}^{(\sigma)}(u^*, p^*) \right) \geq 0,
\]

(6.28)

\[
(\alpha - \alpha^*) \left( D_{\alpha} J_R(\sigma^*, \alpha^*, \psi^*) + 2 \mathcal{G}^{(\alpha)}(u^*, p^*) \right) \geq 0,
\]

(6.29)

\[
D_{\psi} J_R(\sigma^*, \alpha^*, \psi^*), \psi - \psi^* + \left\langle \mathcal{G}(\psi)(u^*, p^*), \psi - \psi^* \right\rangle \geq 0.
\]

(6.30)

**Proof.** We consider a sequence of parameters \(\epsilon_n\) such that

1. \((\sigma_{\epsilon_n}^*, \alpha_{\epsilon_n}^*, \psi_{\epsilon_n}^*, u_{\epsilon_n}^*)\) is a sequence of solutions to (6.4) converging to \((\sigma^*, \alpha^*, \psi^*, u^*)\) as in Lemma 6.1.
2. \(u_{\epsilon_n}^*(t, x) > u_0(x) + \epsilon_n\) for all \((t, x)\) with \(|x - x_{ob}| \leq a\) and \(T - t \leq a\),
3. for the adjoint states \(p_{\epsilon_n}^*\) defined by Lemma 6.2, \(p_{\epsilon_n}^*\) weakly converges to \(p^*\) in \(L^2((0, T) \times (0, X))\) and \(\phi p_{\epsilon_n}^*\) weakly converges to \(\phi p^*\) in \(L^2(0, T; V_X)\), where \(\phi\) is given in Lemma 6.2.

We drop the index \(n\) in \(\epsilon_n\). We have to prove that \(\lim_{\epsilon \to 0} \mathcal{G}(\sigma)(u_{\epsilon}^*, p_{\epsilon}^*) = \mathcal{G}(\sigma)(u^*, p^*)\).

Since \(u_{\epsilon}^* \to u^*\) strongly in \(L^2(0, T; V_X)\) (see Lemma 6.1) and \(\phi p_{\epsilon}^* \to \phi p^*\) weakly in \(L^2(0, T; V_X)\), we deduce that

\[
\lim_{\epsilon \to 0} \int_0^T \left\langle \frac{\partial^2 u_{\epsilon}^*}{\partial x^2}, \phi p_{\epsilon}^* \right\rangle = \int_0^T \left\langle \frac{\partial^2 u^*}{\partial x^2}, \phi p^* \right\rangle.
\]

On the other hand, \((1 - \phi) \frac{\partial^2 u_{\epsilon}^*}{\partial x^2}\) strongly converges to \((1 - \phi) \frac{\partial^2 u^*}{\partial x^2}\) in \(L^2((0, T) \times (0, X))\) from Lemma 6.1, and \(p_{\epsilon}^*\) weakly converges to \(p^*\) in \(L^2((0, T) \times (0, X))\). Thus

\[
\lim_{\epsilon \to 0} \int_0^T \int_X \left( (1 - \phi) \frac{\partial^2 u_{\epsilon}^*}{\partial x^2} \right) p_{\epsilon}^* = \int_0^T \int_X \left( (1 - \phi) \frac{\partial^2 u^*}{\partial x^2} \right) p^*.
\]

From the two points above, we see that \(\lim_{\epsilon \to 0} \mathcal{G}(\sigma)(u_{\epsilon}^*, p_{\epsilon}^*) = \mathcal{G}(\sigma)(u^*, p^*)\); we can pass to the limit in (6.11) and obtain (6.28).

We have to prove that \(\lim_{\epsilon \to 0} \mathcal{G}(\alpha)(u_{\epsilon}^*, p_{\epsilon}^*) = \mathcal{G}(\alpha)(u^*, p^*)\). The fact that \(u_{\epsilon}^* \to u^*\) weakly in \(L^2(0, T; V_X)\) implies
strongly in $L^2(0, T; V_X)$ (see Lemma 6.1) implies that $B_{e, X}^{(\alpha)} u^*_n$ converges to $B_X^{(\alpha)} u^*$ in $L^2(0, T; (V_X')^*$. On the other hand, $\phi p^*_n \to \phi p^*$ weakly in $L^2(0, T; V_X)$. This implies that

$$\lim_{\epsilon \to 0} \int_0^T \left\langle B_{e, X}^{(\alpha)} u^*_n, \phi p^*_n \right\rangle = \int_0^T \left\langle B_X^{(\alpha)} u^*, \phi p^* \right\rangle.$$  

It can also be checked that $(1 - \phi) B_{e, X}^{(\alpha)} u^*_n$ strongly converges to $(1 - \phi) B_X^{(\alpha)} u^*$ in $L^2((0, T) \times (0, X))$. From the weak convergence of $p^*_n$ to $p^*$ in $L^2((0, T) \times (0, X))$, we deduce that

$$\lim_{\epsilon \to 0} \int_0^T \int_0^X \left( (1 - \phi) B_{e, X}^{(\alpha)} u^*_n \right) p^*_n = \int_0^T \int_0^X \left( (1 - \phi) B_X^{(\alpha)} u^* \right) p^*.$$

The two points above yield that $\lim_{\epsilon \to 0} G^{(\alpha)}_e (u^*_n, p^*_n) = G^{(\alpha)} (u^*, p^*)$. This and (6.12) yield (6.29). The last condition (6.30) is obtained in the same manner.  

6.2. Conclusion: optimality condition for the calibration problem.

For calibrating the Lévy process, one observes the spot price $S$ and the prices $(\hat{\bar{u}}_i)_{i \in I}$ of a family of American put options with maturities/strikes given by $(T_i, x_i)$, and we call $\bar{u}_i = \hat{\bar{u}}_i - x_i + S$, $i \in I$. We assume that

$$\bar{u}_i > u_o(x_i), \quad \text{for all } i \in I.$$

Let $\bar{X}$ be such that for all $(\sigma, \alpha, \psi) \in \mathcal{H}$, the exercise price $\gamma(t)$ is smaller than $\bar{X}$ for all $t < T$, and take $X \geq \bar{X}$. The calibration problem has the form (6.3) with the new definition of $J$:

$$J(u) = \sum_{i \in I} \omega_i (u(T_i, x_i) - \bar{u}_i)^2,$$

where $\omega_i$ are positive weights.

As above, we can also define the modified least square problem (6.4), and have $\epsilon$ tend to 0. Let a subsequence $(\sigma_{e, n}, \alpha_{e, n}, \psi_{e, n}, u_{e, n}^*)$ of solutions of (6.4) converge to $(\sigma^*, \alpha^*, \psi^*, u^*)$ as in Lemma 6.1, then $(\sigma^*, \alpha^*, \psi^*, u^*)$ is a solution of (6.3).

We assume that $u^*(T_i, x_i) > u_o(x_i)$, for all $i \in I$. It is clear from the continuity of $u^*$ and from the uniform convergence of $u_{e, n}^*$ that there exists a positive real number $a$ and an integer $N$ such that for $n > N$, $u_{e, n}^*(t, x) > u_o(x) + \epsilon_n$ for all $(t, x)$ such that $|t - T_i| < a$ and $|x - x_i| < a$ for some $i \in I$. We may fix a smooth function $\phi$ taking the value 1 for all $x$ such that $|x - x_i| \geq a/2$, $|T_i - t| \geq a/2$ for all $i \in I$, and vanishing in neighborhoods of $(T_i, x_i)$, $i \in I$.

Calling $A_X$ the operator defined by (4.2), (3.9) and (2.12) with the parameters $(\sigma, \alpha, \psi) = (\sigma^*, \alpha^*, \psi^*)$, we obtain the optimality conditions exactly as in § 6.1.4:

**Theorem 6.6.** Under the assumptions made at the beginning of § 6.2, there exists a function $p^* \in L^2((0, T) \times (0, X))$ and a Radon measure $\xi^*$ such that for all $v \in Z$, $(Z$ is defined by (6.14))

$$\int_0^T \int_0^X \left( \frac{\partial v}{\partial t} + A_X v \right) p^* + <\xi^*, v> = 2 \sum_{i \in I} \omega_i (u^*(T_i, x_i) - \bar{u}_i) v((T_i, x_i)),$$

and (6.17), (6.18), and such that (6.28), (6.29) and (6.30) are satisfied for all $(\sigma, \alpha, \psi) \in \mathcal{H}$, with $G^{(\sigma)}$, $G^{(\alpha)}$ and $G^{(\psi)}$ defined respectively by (6.25), (6.26) and (6.27), (with the new choice of $\phi$).

**Proof.** The proof follows exactly the same lines as that of Proposition 6.5. ♦

7. Appendix 1.
Proof of Lemma 3.2. By the change of variable \( y = \log(x) \),
\[
|v|_{\phi, s}^2 = \int_\mathbb{R} e^{vy} \int_\mathbb{R} \frac{\phi(z)}{|z|^{1+2s}} (v(e^{y^2} - v(\xi)))^2 \, dz = \int_\mathbb{R} dy \int_\mathbb{R} \frac{\phi(z)}{|z|^{1+2s}} (e^{y^2} \hat{v}(y - z) - \hat{v}(y))^2 \, dz.
\]
By Fubini’s theorem, \( |v|_{\phi, s}^2 = \int_\mathbb{R} d\xi |\phi(z)||z|^{-(1+2s)} \int_\mathbb{R} (e^{\xi} \hat{v}(y - z) - \hat{v}(y))^2 \, dy \); after a Fourier transform with respect to the variable \( y \),
\[
|v|_{\phi, s}^2 = \int_\mathbb{R} d\xi \phi(z)||z|^{-(1+2s)} \int_\mathbb{R} e^{\xi(\xi - \xi^2)} \hat{v}(\xi) - \hat{v}(\xi)|^2 \, d\xi
\]
\[
= \int_\mathbb{R} d\xi \phi(z)||z|^{-(1+2s)} \int_\mathbb{R} (e^{\xi} + 1 - 2e^{\xi} \cos(\xi))|\hat{v}(\xi)|^2 \, d\xi
\]
\[
= \int_\mathbb{R} d\xi \phi(z)||z|^{-(1+2s)} \int_\mathbb{R} ((e^{\xi} - 1)^2 + 4e^{\xi} \sin^2(\xi/2))|\hat{v}(\xi)|^2 \, d\xi
\]
\[
= \left( \int_\mathbb{R} \frac{\phi(z)}{|z|^{1+2s}} (e^{\xi} - 1)^2 \right) \|v\|_{L^2(\mathbb{R}_+)}^2 + 4 \int_\mathbb{R} \frac{\phi(z)}{|z|^{1+2s}} \int_\mathbb{R} \sin^2(\xi/2)|\hat{v}(\xi)|^2 \, d\xi.
\]
But \( \int_\mathbb{R} d\xi \frac{\phi(z)e^{\xi}}{|z|^{1+2s}} \int_\mathbb{R} \sin^2(\xi/2)|\hat{v}(\xi)|^2 \, d\xi = 4 \int_\mathbb{R} |\xi|^2 |\hat{v}(\xi)|^2 \int_\mathbb{R} \frac{\phi(z) e^{\xi}}{|z|^{1+2s}} \sin^2(\xi/2) \, d\xi \). Define
\[
C_1 = \int_\mathbb{R} \phi(z)||z|^{-(1+2s)} (e^{\xi} - 1)^2 \, d\xi,
\]
which is a real number since \( z \mapsto \phi(z) \max(e^{\xi}, 1) \) is bounded. Similarly, there exists a constant \( \beta > 0 \) such that
\[
4 \int_\mathbb{R} \frac{\phi(z) e^{\xi}}{|\xi|^{1+2s}} \sin^2(\xi/2) |\xi| \, d\xi \leq 4\beta \int_\mathbb{R} \frac{\sin^2(\xi/2)}{|\xi|^{1+2s}} |\xi| \, d\xi = 4\beta \int_\mathbb{R} \frac{\sin^2(\xi/2)}{|u|^{1+2s}} \, du.
\]
and we introduce \( C_2 = 4\beta \int_\mathbb{R} \frac{\sin^2(\xi/2)}{|u|^{1+2s}} \, du \). We have obtained that
\[
|v|_{\phi, s}^2 \leq C_1 \|v\|_{L^2(\mathbb{R}_+)}^2 + C_2 \int_\mathbb{R} |\xi|^2 |\hat{v}(\xi)|^2 \, d\xi.
\]
On the other hand,
\[
\int_\mathbb{R} \frac{\phi(z) e^{\xi}}{|\xi|^{1+2s}} \sin^2(\xi/2) |\xi| \, d\xi = \int_\mathbb{R} \phi(\frac{u}{|\xi|}) e^{\frac{u}{|\xi|}} \sin^2(\frac{u}{2}) \sin^2(\frac{|\xi|}{2}) \, du \geq \int_{-1}^1 \phi(\frac{u}{|\xi|}) e^{\frac{u}{|\xi|}} \sin^2(\frac{|\xi|}{2}) \, du
\]
implies that
\[
\liminf_{|\xi| \to \infty} \int_\mathbb{R} \phi(z) e^{\xi} |\xi|^{-(1+2s)} \sin^2(\frac{\xi}{2}) |\xi| \, d\xi \geq \phi(0) \int_{-1}^1 \sin^2(\frac{u}{2}) |u|^{-(1+2s)} \, du,
\]
which shows that there exists a constant \( M > 0 \) such that for \( |\xi| \geq M \),
\[
4 \int_\mathbb{R} \phi(z) e^{\xi} |\xi|^{-(1+2s)} \sin^2(\frac{\xi}{2}) |\xi| \, d\xi \geq 2\phi(0) \int_{-1}^1 \sin^2(\frac{u}{2}) |u|^{-(1+2s)} \, du,
\]
and we introduce \( C_4 = 2\phi(0) \int_{-1}^1 \sin^2(\frac{|\xi|}{2}) \, du \). Thus
\[
|v|_{\phi, s}^2 \geq C_1 \|v\|_{L^2(\mathbb{R}_+)}^2 + C_4 \int_{|\xi| > M} |\xi|^2 |\hat{v}(\xi)|^2 \, d\xi \geq \frac{C_1}{2} \|v\|_{L^2(\mathbb{R}_+)}^2 + C_3 \int_{|\xi| > M} |\xi|^2 |\hat{v}(\xi)|^2 \, d\xi,
\]
with \( C_3 = \min(C_1/(2M^{2s}), C_4) \). \( \square \)
Proof of Lemma 3.3. It is enough to prove that $B|_{D(\mathbb{R}^+)}$ is continuous from $D(\mathbb{R}^+)$ endowed with the norm of $V^s$ to $V^{s-\max(2\alpha,1)}$ if $\alpha \neq 1/2$, and to $V^{s-1/2}$ if $\alpha = 1/2$. For that, we use the change of variable $y = \log(x)$ and call $\tilde{u}$ the function defined on $\mathbb{R}$ by $\tilde{u}(y) = u(e^y)\tilde{v}$. This yields that $\langle Bu, v \rangle = \langle \tilde{B}\tilde{u}, \tilde{v} \rangle$, where

$$
\tilde{B}\tilde{u}(y) = -\int_{\mathbb{R}} k(z)e^z \left( (1 - e^{-z}) \left( \frac{\partial}{\partial y} u(y) - \frac{1}{2} \tilde{u}(y) \right) + (e^{\tilde{\Phi}} \tilde{u}(y) - \tilde{u}(y)) \right) \, dz,
$$

$$
= -\int_{\mathbb{R}} k(z)e^z \left( (1 - e^{-z}) \left( \frac{\partial}{\partial y} u(y) + \frac{1}{2} e^{-z} - \frac{3}{2} \right) \tilde{u}(y) + \frac{1}{2} e^{-z} \tilde{u}(y) - \tilde{u}(y) \right) \, dz.
$$

The Fourier transform of $\tilde{B}\tilde{u}$ is

$$
\hat{\tilde{b}} = -\hat{\tilde{u}}(\xi) \int_{\mathbb{R}} k(z) e^z \left( (1 - e^{-z})i\xi + \frac{1}{2} e^{-z} - \frac{3}{2} + e^{\tilde{\Phi}} e^{-i\xi z} \right) \, dz,
$$

We make out three cases:

1) $\alpha > 1/2$. In this case, one sees that

$$(1 - e^{-z})i\xi + \frac{1}{2} e^{-z} - \frac{3}{2} + e^{\tilde{\Phi}} e^{-i\xi z} = \frac{1}{2} (e^{-z} + 2e^{\tilde{\Phi}} - 3) + e^{\tilde{\Phi}} (e^{-i\xi z} - 1 + iz\xi)
- i\xi (e^{-z} - 1 + ze^{\tilde{\Phi}}).$$

Therefore

$$
\hat{\tilde{b}} = -\hat{\tilde{u}}(\xi) \left( \int_{\mathbb{R}} \frac{k(z)}{2} (1 + 2e^{\tilde{\Phi}} - 3e^z) + \int_{\mathbb{R}} k(z)e^{\tilde{\Phi}} (e^{-i\xi z} - 1 + iz\xi) \right),
$$

From the assumption on $\psi$, the first integral is a real number independent of $\xi$. As in Lemma 3.2, by introducing $\theta = \xi/|\xi|$ for $\xi \neq 0$, and writing that

$$\int_{\mathbb{R}} k(z)e^{\tilde{\Phi}} (e^{-i\xi z} - 1 + iz\xi) \, dz = |\xi|^{2\alpha} \int_{\mathbb{R}} \left| y \right|^{-(1+2\alpha)} \psi \left( \frac{y}{|\xi|} \right) e^{\frac{\tilde{\Phi}}{2i\xi} (e^{-i\theta y} - 1 + iy\theta)} \, dy,$$

we see from the assumptions on $\psi$ that there exists a positive constant $C_1$ such that $\left| \int_{\mathbb{R}} k(z)e^{\tilde{\Phi}} (e^{-i\xi z} - 1 + iz\xi) \, dz \right| \leq C_1 |\xi|^{2\alpha}$, because $y \mapsto \psi(y) \exp(3y^2/2)$ is a real bounded function and $\int_{\mathbb{R}} |e^{-iy\theta} - 1 + iy\theta|^d|y|^{-(1+2\alpha)} \, dy$ can be bounded independently of $\xi$. The third integral in (7.1) is a real number independent of $\xi$. Therefore,

$$|\hat{\tilde{b}}| \lesssim (1 + |\xi| + |\xi|^{2\alpha}) |\hat{\tilde{u}}(\xi)| \lesssim (1 + |\xi|^{2\alpha}) |\hat{\tilde{u}}(\xi)|.$$  

2) $\alpha < 1/2$. We can still split $\hat{\tilde{b}}$ as in (7.1). From the assumption on $\psi$, the first integral is a real number independent of $\xi$. The second integral can be split into the sum of $\int_{\mathbb{R}} k(z)e^{\tilde{\Phi}} (e^{-i\xi z} - 1) \, dz$ and $i\xi \int_{\mathbb{R}} k(z)e^{\tilde{\Phi}} zdz$, which are both bounded by $C_1 |\xi|$. The third integral is a real number independent of $\xi$, so

$$|\hat{\tilde{b}}| \lesssim (1 + |\xi|)|\hat{\tilde{u}}(\xi)|.$$  

3) $\alpha = 1/2$. The only change with respect to the previous two cases concerns the second integral: we write it as

$$|\xi| \int_{\mathbb{R}} \psi \left( \frac{y}{|\xi|} \right) e^{\frac{\tilde{\Phi}}{2i\xi} (e^{-iy\theta} - 1 + iy1_{|y|<1}\theta)} \, dy + i\theta |\xi| \int_{|\xi|>1} \psi (z) e^{\frac{\tilde{\Phi}}{2|z|^2}} \, dz, \quad (7.2)$$
The first integral in (7.2) is bounded by a constant independent of $\xi$, because $z \mapsto \psi(z) \exp(3z/2)$ is a bounded function and $\int_{\mathbb{R}} |y|^{-2} e^{-iy\theta} - 1 + iy 1_{|y|<1} \theta$ can be bounded independently of $\xi$. For the second integral in (7.2), we have, if $\xi > 1$

$$\left| \int_{|z|>1} \psi(z) e^{\frac{\bar{z}}{|z|^2}} \frac{z}{|z|^2} \, dz \right| \leq \int_{|z|>1} \psi(z) e^{\frac{\bar{z}}{|z|^2}} \, dz + \int_{|\xi|^{-1} |z| \leq 1} \psi(z) e^{\frac{\bar{z}}{|z|^2}} \, dz \lesssim (1 + \log(\xi))$$

$$|\hat{\theta}| \lesssim (1 + |\xi| + |\xi \log(|\xi|))|\hat{u}(\xi)|. \quad \Box$$

**Proof of Lemma 3.5.** It is enough to prove the result for $u, v \in D(\mathbb{R}_+)$:

$$\langle Bu, v \rangle = -\int_{\mathbb{R}_+} dx \int_{\mathbb{R}} k(z) \left( x(e^z - 1) \frac{\partial u}{\partial x}(x) + e^z (u(xe^{-z}) - u(x)) \right) v(x) \, dz = I + II + III,$$

where

$$I = \int_{\mathbb{R}_+} dx \int_{\mathbb{R}} k(z) e^z (u(x) - u(xe^{-z}))(v(x) - v(xe^{-z})) \, dz,$$

$$II = -\int_{\mathbb{R}_+} dx \int_{\mathbb{R}} k(z) x(e^z - 1) \frac{\partial u}{\partial x}(x) v(x) \, dz,$$

$$III = \int_{\mathbb{R}_+} dx \int_{\mathbb{R}} k(z) e^z (u(x) - u(xe^{-z})) v(xe^{-z}) \, dz.$$

But

$$II = \int_{\mathbb{R}_+} dx \int_{\mathbb{R}} k(z) x(e^z - 1) \frac{\partial u}{\partial x}(x) u(x) \, dz + \int_{\mathbb{R}_+} dx \int_{\mathbb{R}} k(z) (e^z - 1) u(x) v(x) \, dz.$$

From this,

$$II + III = \int_{\mathbb{R}_+} \left( \int_{\mathbb{R}} k(z) \left( x(e^z - 1) \frac{\partial u}{\partial x}(x) + e^z (v(xe^{-z}) - v(x)) \right) \, dz \right) u(x) \, dx$$

$$+ \left( \int_{\mathbb{R}} k(z)(2e^{z} - e^{2z} - 1) \, dz \right) \int_{\mathbb{R}_+} u(x) v(x) \, dx.$$

The desired result is obtained. \Box

**Proof of Lemma 3.6.** The assertion is already proved in the case $\alpha < 1/2$, thanks to Lemma 3.3 and Remark 4. Thus, let us focus on the case when $\alpha \geq 1/2$: after a few calculations, one sees that

$$(B^T u - Bu)(x) = \int_{\mathbb{R}} k(z) \left( x(e^z - 1)(2 \frac{\partial u}{\partial x}(x) + u(x)) + e^z u(xe^{-z}) - e^{2z} u(xe^{-z}) \right) \, dz.$$ 

The same change of variables as in the proof of Lemma 3.3 leads to $\langle (B - B^T)u, v \rangle = \langle (B - B^T)\tilde{u}, \tilde{v} \rangle$ where

$$(B\tilde{u} - B^T\tilde{u})(y) = -\int_{\mathbb{R}} k(z) \left( 2(e^z - 1) \frac{\partial \tilde{u}}{\partial y}(y) + e^{\frac{z}{2}} (\tilde{u}(y - z) - \tilde{u}(y + z)) \right) \, dz.$$ 

The Fourier transform of $(\hat{B} - \hat{B^T})\hat{u}$ is

$$-\hat{u}(\xi) \int_{\mathbb{R}} k(z) \left( 2i\xi(e^z - 1) - e^{\frac{z}{2}} (e^{i\xi z} - e^{-i\xi z}) \right) \, dz$$

$$= -2i\xi \hat{u}(\xi) \int_{\mathbb{R}} k(z)(e^z - 1 - ez^{\frac{1}{2}}) \, dz + \hat{u}(\xi) \int_{\mathbb{R}} k(z) e^{\frac{z}{2}} (e^{i\xi z} - e^{-i\xi z} - 2i\xi z) \, dz.$$
From the assumptions, the first integral in the sum above is a real number. Let us focus on the second integral: since the function $z \mapsto e^{i\xi z} - e^{-i\xi z} - 2i\xi z$ is odd, $\psi(0) \int_{\mathbb{R}} |z|^{-(1+2\alpha)}e^{-\frac{|z|^2}{2}} (e^{i\xi z} - e^{-i\xi z} - 2i\xi z) \, dz = 0$, and

$$\int_{\mathbb{R}} k(z)e^{\frac{1}{2}} (e^{i\xi z} - e^{-i\xi z} - 2i\xi z) \, dz = \int_{\mathbb{R}} |z|^{-(1+2\alpha)}z\omega(z) (e^{i\xi z} - e^{-i\xi z} - 2i\xi z) \, dz$$

$$\leq \int_{\mathbb{R}} |z|^{-2\alpha}e^{-\zeta|z|} (e^{i\xi z} - e^{-i\xi z} - 2i\xi z) \, dz \lesssim |\xi|^{2\alpha-1} \lesssim (1 + |\xi|),$$

where, in the case $\alpha = 1/2$, we have used the fact that $|\sin(\xi z)|/|z| \leq |\xi|$. This concludes the proof. □

8. Appendix 2.

**Proof of Proposition 4.1.** Consider $X'$, $0 < X'< X$ and let $\phi$ be a smooth cut-off function taking the value 1 in $[0, 3/4X'+ \frac{1}{4}X]$ and 0 in $[3/4X + \frac{1}{4}X', X]$. It is a possible to prove that $A_X(\mathcal{E}_X(\phi v)) \in L^2((\mathbb{R}^+))$, which yields that $\mathcal{E}_X(\phi v) \in V^2$ and $\phi v \in W^3_X$. This yields the first statement of Proposition 4.1.

Assume that $v \in V_X$ is such that $A_X v \in L^2((0, X))$. Then there exists $f \in L^2((0, X))$ such that

$$-\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} = f - B_X v.$$  \hspace{1cm} (8.1)

If $0 \leq \alpha < 1/2$, then, from Lemma 3.3, $B_X v \in L^2((0, X))$, and (8.1) implies that $v \in W^2_X \cap V_X$.

If $\alpha > 1/2$, then, from Lemma 3.3, $B_X v \in V^{1-2\alpha}_X$. From this and (8.1), one immediately deduces that $v \in V_X \cap W^{3-2\alpha}_X$. A boot-strap argument is needed for improving this result:

if $1/2 < \alpha < 3/4$, then for all $\epsilon > 0$, $v \in W^{3/2-\epsilon}_X$, and $\mathcal{E}_X(v) \in V^{3/2-\epsilon}$. Note that we cannot give a better regularity result for $\mathcal{E}_X(v)$, for example $\mathcal{E}_X(v) \in V^{3/2+\epsilon'}$, because this would require the condition $\partial x^2(x = X) = 0$, which is not proved. Then Lemma 3.3 yields that $B_X v \in L^2((0, X))$, and that $v \in W^2_X \cap V_X$ from (8.1). In the case $\alpha = 1/2$, we obtain from Lemma 3.3 that $v \in W^2_X \cap V_X$ as well. If $\alpha = 3/4$, the same argument shows that $v \in W^{3/2-\epsilon}_X \cap V_X$ for all $\epsilon > 0$.

On the contrary, if $3/4 < \alpha < 1$, we have to keep on boot-strapping: $v \in W^3_X \cap W^{3-2\alpha}_X$ implies that $B_X v \in V^{3-4\alpha}$, and from (8.1), $v \in W^{3-4\alpha}_X$. Either $3/4 < \alpha < 7/8$, and we see that there exists $\epsilon > 0$ such that $v \in W^{3/2+\epsilon}_X$, or $7/8 < \alpha < 1$, and we keep on boot-strapping. After a finite number of steps, we obtain the first two statements of Proposition 4.1.

Then we obtain that $\frac{\partial v}{\partial z} \in C^0((0, X))$ from Sobolev imbeddings. □

**Proof of Theorem 4.2.** For brevity, and since the proof uses rather classical arguments, we shall omit some details. By using results on parabolic equations with monotone operators [26] page 156, it is possible to prove that (4.6) has a unique weak solution in $L^2(0; T; V_X) \cap C^0([0, T]; L^2(0, X)))$, with $\frac{\partial x}{\partial t} \in L^2(0; T; V_X')$. Note that for all $t_0$, $0 < t_0 < T$, $u_{X, \epsilon}$ is smooth in $(t_0, T) \times [a, b]$, where $[a, b]$ is any interval strictly contained in $(0, S)$ or in $(S, X)$, w. From (3.11), the weak maximum principle may be used. It yields that, almost everywhere, $u_{X, \epsilon}$ is nonnegative on the one hand, and greater than or equal to $x$ on the other hand. Therefore, for almost every time $t$, $u_{X, \epsilon}(t) \in K_X$. This implies that $0 \leq \rho x(1 - \{x > S\}) \leq \rho x(1 - \{x > S\}) \leq \rho x$.

From this and (4.6), $u_{X, \epsilon}$ belongs to $C^0([0, T]; V_X) \cap L^2(0, T; D_X)$, $\frac{\partial u_{X, \epsilon}}{\partial t} \in L^2([0, T] \times (0, X))$ and the norms $\|u_{X, \epsilon}\|_{L^2(0, T; V_X)}$, $\|u_{X, \epsilon}\|_{L^2(0, T; D_X)}$, $\|\partial x u_{X, \epsilon}\|_{L^2((0, T) \times (0, X))}$ are bounded independently of $\epsilon$.

Since $V \subset C^0((0, +\infty))$ and since for any $t$, $\lim_{\epsilon \to 0} u_{X, \epsilon}(t) = S$ (because $S - x \leq 27$,
Since $y$ is independent of $u$, we depend of $X$ (4.6) and by observing that the constants in Gårding's inequality for $V_t$, we also know that $X, \epsilon > \infty$.

Proof of Theorem 4.3. We know that $u_{X, \epsilon}$ belongs to $C^0([\tau, T]; D_X)$ for all $\tau$, $0 < \tau < T$. Therefore, from Proposition 4.1, $u_{X, \epsilon} \in C^0([\tau, T]; W_X^{3/2+\epsilon})$ for some positive $\epsilon$. This yields that for each time $t > 0$, $u_{X, \epsilon} \in C^1((0, X])$. On the other hand, we know that $u_{X, \epsilon}(t, X) = 0$ for $t \in [0, T]$, and $u_{X, \epsilon} \geq 0$ in $[0, T] \times [0, X]$. From the last three observations, we see that for all $t$, $0 < t < T$, $\frac{\partial u}{\partial x}(t, x) \leq 0$. We aim at proving that for each $t > 0$ there exists a number $\epsilon(t) < X$, such that $\frac{\partial u}{\partial x}(t, x) = 0$ if $\epsilon(t) < x < X$. Indeed, it it was not case, we would be in one of the following two cases:

1) $\frac{\partial u}{\partial x}(t, y_n) > 0$ in some interval $[y(t), X]$, $y(t) < X$. This implies that $u_{X, \epsilon}(t, x) < 0$ in $(y(t), X)$, which is impossible since $u(t, x) \geq 0$.

2) There exists a strictly increasing sequence of numbers $y_n$, $0 < y_n < y_{n+1} < X$, such that $\lim_{n \to \infty} y_n = X$ and $\frac{\partial u}{\partial x}(t, y_n) = 0$, $\frac{\partial u}{\partial x}(t, x) > 0$ for $x$ in $(y_{2n}, y_{2n+1})$, and negative for $x$ in $(y_{2n+1}, y_{2n+2})$. The numbers $y_{2n}$, $n \in \mathbb{N}$ are local minima of $u_{X, \epsilon}(t, x)$. Let us consider the terms entering equation (4.6) at $x = y_{2n}$: we have $\frac{\partial u}{\partial x}(t, y_{2n}) \geq 0$ and $\lim_{n \to \infty} \frac{\partial u}{\partial x}(t, y_{2n}) = 0$. It is clear that $-\frac{\partial^2}{\partial x^2} u_{X, \epsilon}(t, y_{2n}) \leq 0$ and $\lim_{n \to \infty} \frac{\partial u}{\partial x}(t, y_{2n}) = 0$ because $y_{2n}$ is a local minimum. We also know that $r_{y_{2n}}(1 - 1_{(y_{2n}, y_{2n+1})}) \frac{\partial u}{\partial x}(t, y_{2n})) \geq 0$ and that $\lim_{n \to \infty} r_{y_{2n}}(1 - 1_{(y_{2n}, y_{2n+1})}) \frac{\partial u}{\partial x}(t, y_{2n})) = 0$. Therefore

$$\lim_{n \to \infty} B_X u_{X, \epsilon}(t, y_{2n}) \geq 0,$$

and, since $\frac{\partial u}{\partial x}(t, y_{2n}) = 0$,

$$\lim_{n \to \infty} \sup_{t} \int_R k(z) \left(e^z \frac{\partial x}{\partial x}(t, y_{2n}e^{-z}) - u_{X, \epsilon}(t, y_{2n})\right) dz \leq 0.$$

This yields $\int_0^\infty k(z)e^z u_{X, \epsilon}(t, X e^{-z}) dz \leq 0$, which is impossible since $u_{X, \epsilon} \geq u_0$.

Therefore, for all $t > 0$, the function $x \mapsto \frac{\partial u}{\partial x}(t, x)$ is nonpositive in a neighborhood of $X$, and $(\frac{\partial u}{\partial x}(t, x))_+$ is zero near $X$. 

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Moreover, since \( u_{X, \varepsilon} \) is nondecreasing and \( u_{X, \varepsilon}(\cdot, X) = 0 \), the function \( t \mapsto \frac{\partial u_{X, \varepsilon}}{\partial x}(t, X) \) is nonincreasing. Therefore, there exists \( \tau_0, 0 \leq \tau_0 \leq T \), such that \( \frac{\partial u_{X, \varepsilon}}{\partial x}(t, X) < 0 \) for \( \tau_0 < t \leq T \) and \( \frac{\partial u_{X, \varepsilon}}{\partial x}(t, X) = 0 \) for \( 0 \leq t \leq \tau_0 \).

By taking the derivative of (4.6) with respect to \( x \) and multiplying by \( x \), we see that \( z_{X, \varepsilon} = x \frac{\partial u_{X, \varepsilon}}{\partial x} \) satisfies

\[
\frac{\partial z_{X, \varepsilon}}{\partial t} + A_X z_{X, \varepsilon} - rx 1_{\{x > S\}} V_\varepsilon'(u_{X, \varepsilon}) z_{X, \varepsilon}
= -rx(1 - 1_{x > S} V_\varepsilon'(u_{X, \varepsilon})) + r S V_\varepsilon'(u_{X, \varepsilon}) \delta_{x=S}, \quad t \in (0, T], \; 0 < x < X, \quad (8.4)
\]

\[
z_{X, \varepsilon}(t = 0, x) = -x 1_{0 < x < S}, \quad 0 < x < X.
\]

Since \( z_{X, \varepsilon}(t, X) = 0 \) for \( t \in [0, \tau_0) \), \( \tau_0 \) is defined above, the function \( z_{X, \varepsilon}|_{t \in (0, \tau_0)} \in L^2(0, \tau_0; V_X) \). On the other hand, \( z_{X, \varepsilon}(t, .) \notin V_X \), for \( t \in (\tau_0, T] \). In (8.4), for \( t > \tau_0 \), \( A_X z_{X, \varepsilon}(t) \) i.e.

\[
A_X z_{X, \varepsilon}(t, x) = -\sigma^2 x^2 \frac{\partial^2 z_{X, \varepsilon}}{\partial x^2}(t, x) + rx \frac{\partial z_{X, \varepsilon}}{\partial x}(t, x)
- \int_{\mathbb{R}} k(z) \left( x(e^z - 1) \frac{\partial z_{X, \varepsilon}}{\partial x}(t, x) + e^z(1_{\{x > -\log(x/\varepsilon)\}}) z_{X, \varepsilon}(t, x e^{-z}) - z_{X, \varepsilon}(t, x) \right) dz,
\]

has a sense as a distribution and for all \( X' < X \), belongs to the dual of \( \{v \in V_X, v = 0 \text{ in } (X', X)\} \).

We split the function \( z_{X, \varepsilon} \) into the sum of two functions \( \hat{z}_{X, \varepsilon} \in C^0([0, T]; L^2(0, X)) \) and \( \hat{z}_{X, \varepsilon} \in L^2(0, T; V_X) \) which satisfy

\[
\frac{\partial \hat{z}_{X, \varepsilon}}{\partial t} + A_X \hat{z}_{X, \varepsilon} - rx 1_{\{x > S\}} V_\varepsilon'(u_{X, \varepsilon}) \hat{z}_{X, \varepsilon} = r S V_\varepsilon'(u_{X, \varepsilon}) \delta_{x=S},
\hat{z}_{X, \varepsilon}(t = 0, x) = 0, \quad 0 < x < X,
\]

\[
\hat{z}_{X, \varepsilon}(t, X) = 0, \quad 0 < t < T,
\]

and

\[
\frac{\partial \hat{z}_{X, \varepsilon}}{\partial t} + A_X \hat{z}_{X, \varepsilon} - rx 1_{\{x > S\}} V_\varepsilon'(u_{X, \varepsilon}) \hat{z}_{X, \varepsilon} = -rx(1 - 1_{x > S} V_\varepsilon'(u_{X, \varepsilon}))
\hat{z}_{X, \varepsilon}(t = 0, x) = -x 1_{0 < x < S}, \quad 0 < x < X,
\]

\[
\hat{z}_{X, \varepsilon}(t, X) \leq 0, \quad 0 < t < T.
\]

From the fact that \( u_{X, \varepsilon} \geq u_X^{(L)} \) and from (4.7), we know that

\[
\lim_{\varepsilon \to 0} \|V_\varepsilon(u_{X, \varepsilon}(S))\delta_{x=S}\|_{L^2(0, T; V')} = 0.
\]

Thus, \( \lim_{\varepsilon \to 0} \|\hat{z}_{X, \varepsilon}\|_{L^2(0, T; V')} = 0 \). One can also prove that \( \hat{z}_{X, \varepsilon}(t, 0) = 0 \) and that \( \hat{z}_{X, \varepsilon}(t, 0) \geq 0 \).

From the last observation and since \( z_{X, \varepsilon}(t, 0) = 0 \), we see that for all \( t \in [0, T] \), \( \hat{z}_{X, \varepsilon}(t, 0) = 0 \).

We know that \( \hat{z}_{X, \varepsilon}|_{t \in [0, \tau_0)} \in L^2(0, \tau_0; V_X) \): in \( (0, \tau_0) \times (0, X) \), we can take \( e^{-MT} (\hat{z}_{X, \varepsilon})_+ \) as a test-function in the equation satisfied by \( \hat{z}_{X, \varepsilon} \). From Gårding’s inequality, choosing \( M \) large enough yields that \( \hat{z}_{X, \varepsilon}(t, \cdot)_+ = 0 \) for \( t \in [0, \tau_0] \).

On the other hand, for \( \tau_1 > \tau_0 \), there exists a constant \( \underline{Z} > 0 \) such that \( \hat{z}_{X, \varepsilon}(t, X) \leq -\underline{Z} \) for \( t \in [\tau_1, T] \). This and the continuity of \( \hat{z}_{X, \varepsilon} \), imply that there exits \( X_{\tau_1} \), \( S < X_{\tau_1} < X \), such that \( \hat{z}_{X, \varepsilon} \leq 0 \) in \( [\tau_1, T] \times [X_{\tau_1}, X] \). Therefore, in the time interval \( [\tau_1, T] \), we can take \( (\hat{z}_{X, \varepsilon}(t, x))_+ e^{-MT} \) as a test-function in the equation satisfied by \( \hat{z}_{X, \varepsilon} \), even if \( \hat{z}_{X, \varepsilon} \) does not belong to \( V_X \) (indeed \( (\hat{z}_{X, \varepsilon}(t, \cdot))_+ \) does not see the singular behavior of \( z_{X, \varepsilon}(t, .) \) near \( X \)). From Gårding’s inequality, we have that for \( M \) large enough, \( t \mapsto e^{-MT} \int_0^X ((\hat{z}_{X, \varepsilon})_+)^2(t, x)dx \) is nonincreasing in \( (\tau_1, T) \).
We can have \( \tau_1 \) tend to \( \tau_0 \). This yields that \((\tilde{z}_{X,\varepsilon})_+ = 0 \) in \((\tau_0, T) \times (0, X)\). We have proved that \( \tilde{z}_{X,\varepsilon} \leq 0 \) in \((0, T) \times (0, X)\).

Finally, let \( X \) and \( X' \) be two numbers such that \( S < X < X' \). Call \( \tilde{u}_{X,\varepsilon} \) the function obtained by extending \( u_{X,\varepsilon} \) by \( 0 \) in \([0, T] \times [X, X']\). Clearly, \( \tilde{u}_{X,\varepsilon} \in C^0([0, T]; K_{X'})\). It can be seen from (4.6) and from brevity, we shall omit some details. We call \( Q \) weak solution of obtained by extending \( u_{X,\varepsilon} \) by \( 0 \) in \([0, T] \times (0, X)\). We know that \( \tilde{u}_{X,\varepsilon} \) is continuous. Moreover, there exists a positive constant \( C \) independent of \( \varepsilon \), such that

\[
\sup_{q \in L^2((0, T) \times (0, X))} \inf_{v \in Z_e} \frac{\mathcal{Q}_e (q, v)}{\|q\|_{L^2((0, T) \times (0, X))} \|v\|_{Z_e}} \geq c.
\]

To prove this inf-sup condition, take \( v \in L^2(0, T; V_X) \cap H^1(0, T; L^2((0, X))) \) as the weak solution of

\[
\frac{\partial v}{\partial t} + A_{e,X} v - r x 1_{\{x > S \}} V_e'(u_e) v = q \quad t > 0, \quad v(0, \cdot) = 0.
\]

and observe that \( \|v\|_{Z_e} \leq C \|q\|_{L^2((0, T) \times (0, X))} \) for a constant \( C \) independent of \( \varepsilon \).

Therefore, calling \( Q_e \) the linear and continuous operator from \( L^2((0, T) \times (0, X)) \)
to the dual of $Z$, defined by $<Q_\epsilon p, v> = Q_\epsilon(p, v)$, the range of $Q_\epsilon$ is closed and $Q_\epsilon$ is injective. On the other hand, $Q_\epsilon(q, v)$ for all $q \in L^2((0, T) \times (0, X))$ implies that $v = 0$. Therefore, $Q^*_{T'}$ is injective. We have proved that $Q_\epsilon$ is an isomorphism from $L^2((0, T) \times (0, X))$ onto the dual of $Z$, and that its inverse is continuous with a norm independent of $\epsilon$.

From this and since $z \mapsto 2(u^*_\epsilon(T, x_{ob}) - \bar{u})z((T, x_{ob}))$ is a continuous linear form on $Z$, with a continuity constant independent of $\epsilon$, there exists a unique $p^*_\epsilon \in L^2((0, T) \times (0, X))$ such that for all $v \in Z$, $Q_\epsilon(p^*_\epsilon, v) = 2(u^*_\epsilon(T, x_{ob}) - \bar{u})v((T, x_{ob}))$ and $\|p^*_\epsilon\|_{L^2((0,T) \times (0,X))}$ is bounded independently of $\epsilon$. The first part of the lemma is proved.

To prove the second part of the lemma, consider $G_\epsilon \in L^2((0, T) \times \mathbb{R}_+)$ the solution of the backward Cauchy problem:

$$
\frac{\partial G_\epsilon}{\partial t} + \frac{(\sigma^2_\epsilon)^2}{2} \frac{\partial^2}{\partial x^2} (x^2 G_\epsilon) - B^T_\epsilon G_\epsilon + \frac{\partial}{\partial x}(rx G_\epsilon) = 0, \quad (t, x) \in [0, T) \times \mathbb{R}_+, \quad (8.7)
$$

$$
G_\epsilon(t = T) = -2(u^*_\epsilon(T, x_{ob}) - \bar{u})\delta_{x = x_{ob}},
$$

where $B^T_\epsilon$ is given by (3.6). One can check that $G_\epsilon$ is smooth for $t < T$ and that for any integer $k$ and for any compact $\omega$ in $[0, T] \times [0, +\infty)$ which does not contain $(T, x_{ob})$, the norm of $G_\epsilon$ in $C^k(\omega)$ is bounded independently of $\epsilon$. Also, for the function $\phi$ defined in Lemma 6.2, $\phi G_\epsilon \in L^2(0, T; V)$, with a norm bounded by a constant independent of $\epsilon$.

Let $\chi$ be a smooth function with a compact support contained in $(0, T) \times (0, X)$, taking the value 1 in a neighborhood of $(T, x_{ob})$, and whose support does not intersect the support of $V_\epsilon(u^*_\epsilon)$ for all $\epsilon$. For example, $\chi = 1 - \phi$ can be chosen. With $A^T_{\epsilon, X}$ defined in Remark 11, it may be checked that $\|\chi A^T_\epsilon G_\epsilon - A^T_{\epsilon, X}(\chi G_\epsilon)\|_{L^2((0,T) \times (0,X))}$ is bounded independently of $\epsilon$. The reason for that is that $\chi$ is constant near the point where $G_\epsilon$ is singular.

One sees that $q^*_\epsilon = p^*_\epsilon - \chi G_\epsilon$ is the unique solution (in the very weak sense defined above, i.e. by duality with the functions in $Z'$) of a boundary value problem in $(0, T) \times (0, X)$, of the form

$$
\frac{\partial q^*_\epsilon}{\partial t} - A^T_{\epsilon, X}q^*_\epsilon + rx 1_{\{x > s\}} V^*_\epsilon(u^*_\epsilon)q^*_\epsilon = g^*_\epsilon, \quad (t, x) \in [0, T) \times (0, X),
$$

$$
q^*_\epsilon(t, X) = 0, \quad t \in (0, T),
$$

$$
q^*_\epsilon(T, x) = 0, \quad x \in (0, X),
$$

where $g^*_\epsilon = \frac{\partial}{\partial t} G_\epsilon + \chi A^T_\epsilon G_\epsilon - A^T_{\epsilon, X}(\chi G_\epsilon) \in L^2((0, T) \times (0, X))$. This last boundary value problem has a unique weak solution in $L^2(0, T; V_X)$, with a norm bounded independently of $\epsilon$. The weak and the very weak solutions coincide. Therefore $p^*_\epsilon - \chi G_\epsilon \in L^2(0, T; V_X)$ and $\|p^*_\epsilon - \chi G_\epsilon\|_{L^2(0,T; V_X)}$ is bounded by a constant independent of $\epsilon$. Therefore $\phi p^*_\epsilon \in L^2(0, T; V_X)$, with a norm bounded independently of $\epsilon$. \[\square\]

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