AN INVERSE PROBLEM FOR A PARABOLIC VARIATIONAL INEQUALITY WITH AN INTEGRO-DIFFERENTIAL OPERATOR

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Abstract. We consider the calibration of a Lévy process with American vanilla options. The price of an American vanilla option as a function of the maturity and the strike satisfies a forward in time linear complementarity problem involving a partial integro-differential operator. It leads to a variational inequality in a suitable weighted Sobolev space. Calibrating the Lévy process amounts to solving an inverse problem where the state variable satisfies the previously mentioned variational inequality. We propose a regularized least square method. After studying the variational inequality carefully, we find necessary optimality conditions for the least square problem. In this work, we focus on the case when the volatility is bounded away from zero.

1. Introduction. Consider an arbitrage-free market described by a probability measure $\mathbb{P}$ on a scenario space $(\Omega, \mathcal{A})$. There is a risk-free asset whose price at time τ is $e^{r\tau}$, $r \geq 0$ and a risky asset whose price at time τ is S_τ. Specifying an arbitrage-free option pricing model necessitates the choice of a risk-neutral measure, *i.e.* a probability $\mathbb{P}^*$ equivalent to $\mathbb{P}$ such that the discounted price $(e^{-r\tau}S_\tau)_{\tau\in[0,T]}$ is a martingale under $\mathbb{P}^*$. Such a probability measure $\mathbb{P}^*$ allows for the pricing of European options; consider a European option with payoff $\overline{P_\circ}$ at maturity $t \leq T$: its price at time $\tau \leq t$ is $P_\tau = e^{-r(t-\tau)}\mathbb{E}^{\mathbb{P}^*}(\overline{P_\circ}(S_t)|\mathcal{F}_\tau)$, where $(\mathcal{F}_\tau)_{\tau\in[0,T]}$ is the natural filtration. Similarly, consider an American option with payoff $\overline{P_\circ}$ and maturity $t \leq T$: the price of this option at time τ is

$$P_\tau = \sup_{s\in\mathcal{T}_{\tau,t}} \mathbb{E}^{\mathbb{P}^*}\left(e^{-r(s-\tau)}\overline{P_\circ}(S_s) \middle| \mathcal{F}_\tau \right), \tag{1.1}$$

where $\mathcal{T}_{\tau,t}$ denotes the set of stopping times in $[\tau, t]$.
The pricing model $\mathbb{P}^*$ must be compatible with the prices of the options observed on the market, whose number may be large. *Model calibration* consists of finding $\mathbb{P}^*$ such that the discounted price $(e^{-r\tau}S_\tau)_{\tau\in[0,T]}$ is a martingale, and s.t. the option prices computed by e.g. (1.1) in the case of American options coincide with the observed option prices. This is an *inverse problem.* We focus on the case when the observed prices $(\bar{p}_i)_{i\in I}$ are those of a family of American vanilla put options indexed by $i \in I$, with maturities t_i, (assuming for simplicity $T = \max_{i\in I} t_i$) and strikes x_i.

The Black-Scholes model assumes that $(S_\tau)_{\tau\in[0,T]}$ is a geometric Brownian motion under $\mathbb{P}^*$: $dS_\tau = S_\tau(rd\tau + \sigma dW_\tau)$, where the volatility σ is a constant. Unfortunately, this model is often too simple to match the observed option prices and must be replaced by more involved models:
1) Black-Scholes models with local volatility: the volatility is assumed to be a function of time and of the price of the underlying asset. This volatility function is calibrated by observing the option prices available on the markets and solving inverse problems involving either partial differential equations or inequalities, see[3, 7, 24] for volatility calibration with European options and [2, 5] with American options;
2) models where the volatility is also a stochastic process, see e.g. [21]. The option price is then found as a function of time, the price of the underlying asset and the volatility. These models also lead to parabolic partial differential equations or inequalities with possible degeneracies when the volatility vanishes; stochastic volatility calibration has been performed in [33];
3) models with Lévy driven underlying assets: Lévy processes are processes with stationary and independent increments which are continuous in probability, see the

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book by Cont and Tankov [13] and the references therein, for example [19, 20, 28]. The option price is found by solving partial integro-differential equations or inequalities. Calibration of Lévy models with European options has been discussed in [14, 15]. The present work is devoted to the calibration of Lévy processes with American options. At this stage, it is not yet necessary to discuss Lévy processes in detail. For the moment, we just assume that the model is characterized by parameters θ in a suitable class Θ.

The last two classes of models describe incomplete markets: the knowledge of the historical price process alone does not allow to compute the option prices in a unique manner. When the option prices do not determine the model completely, additional information may be introduced into the problem by specifying a *prior* model. If the historical price process has been estimated statistically from the time series of the underlying asset, this knowledge has to be injected in the inverse problem; calling $\mathbb{P}_0$ the prior probability measure obtained as an estimation of $\mathbb{P}$, we are going to focus on least-square formulations of the type: find $\theta \in \Theta$ which minimizes

$$\sum_{i\in I} \omega_i \left(P^\theta(0, S_\circ, t_i, x_i) - \overline{p_i} \right)^2 + \rho J_2(\mathbb{P}^\theta, \mathbb{P}_0), \tag{1.2}$$

where

- ω_i are suitable positive weights,
- $S_\circ$ is the price of the underlying asset today,
- $P^\theta(0, S_\circ, t_i, x_i)$ is the price of the option with maturity t_i strike x_i, computed with the pricing model associated with θ,
- $\rho J_2(\mathbb{P}^\theta, \mathbb{P}_0)$ is a regularization term which measures the closedness of the model $\mathbb{P}^\theta$ to the prior. The number $\rho > 0$ is called the regularization parameter. This functional has two roles: 1) it stabilizes the inverse problem; for that, ρ should be large enough and J_2 should be convex or at least convex in a large enough region; 2) it guarantees that $\mathbb{P}^\theta$ remains close to $\mathbb{P}_0$ in some sense. The choice of J_2 is very important: $J_2(\mathbb{P}^\theta, \mathbb{P}_0)$ is often chosen as the relative entropy of the pricing measure $\mathbb{P}^\theta$ with respect to the prior model $\mathbb{P}_0$, see [8], because the relative entropy becomes infinite if $\mathbb{P}^\theta$ is not equivalent to $\mathbb{P}_0$. Some authors have argued that such a choice may be too conservative in some cases, for two reasons: a) the historical data which determine the prior may be missing or partially avalaible -b) in the context of e.g. volatility calibration, once the volatility is specified under $\mathbb{P}_0$, then the volatility under $\mathbb{P}^\theta$ must be the same for the relative entropy to be finite. In [9] a different approach was considered which allowed for volatility calibration.

Note that evaluating the functional in (1.2) requires solving $\#I$ linear complementarity problems (LCP for brevity) involving partial integro-differential (PID) operators in the variables τ and S, see §2.1 below. This approach was chosen in [2, 5] for calibrating local volatility with American options.

In the present case (Lévy driven assets), we show that there is a better approach which consists of computing the prices $P^\theta(0, S_\circ, t_i, x_i)$, $i \in I$, by using a single forward in time LCP with a PID operator in the variables maturity t and strike x. This LCP is introduced in § 2.2, see (2.9-2.11) below. It is reminiscent of the forward equation (known as Dupire's equation in the finance community) which is often used for local volatility calibration with vanilla European options, see [4, 18]. We then find a new least square problem, where the functional is evaluated by solving a single LCP involving a PID operator. The main goal of the paper is to study this least square problem theoretically for a rather general parameterization of the Lévy density k, see (2.14) below, with the volatility σ bounded away from 0, and to give necessary optimality conditions. This problem has connections with some optimal control problems for variational inequalities studied in [11, 23, 32]. The article of Hintermüller [22] on an inverse problem for an elliptic variational inequality has inspired [2] and the present work.

As far as we know, this is the first attempt at calibrating Lévy processes with American options, so comparison with other methods is difficult. The results below can be used in practice because they have their discrete counterparts when finite elements or finite differences are used. The accuracy is expected to be similar to the one observed in [5].

The paper is organized as follows. In §2, we obtain the forward LCP (2.9-2.11) and make some assumptions on the Lévy density. In § 3, we introduce a family of fractional weighted Sobolev spaces and give preliminary results on the nonlocal operator in (2.9). In § 4 we carefully study the variational inequality stemming from (2.9)-(2.11). For the analysis, we must first study a regularized nonlinear problem posed in a bounded domain, then let the regularization parameter tend to 0 and the domain's boundary tend to infinity. The sensitivity of the solution to variations of σ and k is discussed in § 5. Finally, the inverse problem is studied in § 6: necessary optimality conditions are given. Some technical proofs are postponed to §7 and 8.

For the reader's convenience, let us point out the main results of this work:

• the forward complementarity problem is written in (2.9)-(2.11) and the assumptions on k are described in § 2.3.

• Theorem 4.9 contains a result of existence and uniqueness for the variational inequality associated to (2.9)-(2.11) in suitable Sobolev spaces. It is also proved that the related free boundary stays in a bounded region. Note that by using the theory presented in [10], it is possible to study the variational inequality in Sobolev spaces with decaying weights as $x \to 0$ and $x \to +\infty$, (actually the variable $\log(x)$ was used instead of x in [10]). Here, we show that these weights can be avoided. Another advantage of the present analysis is that it can be extended to the case when $\sigma = 0$ by singular perturbation arguments, if the Lévy measure is chosen to keep the problem parabolic. This will be done in a forthcoming work, [1].

• The sensitivity of solutions w.r.t. variations of the Lévy process is studied in § 5.

• Theorem 6.6 contains the necessary optimality conditions for the least square inverse problem. These conditions are obtained by first studying a modified inverse problem whose state variable satisfies the above mentioned regularized nonlinear problem, then by passing to the limit as the regularization parameter tends to zero.

2. Description of the model.

2.1. The backward linear complementarity problem. For a Lévy process $(X_\tau)_{\tau>0}$ on a filtered probability space, the Lévy-Khintchine formula says that there exists a function $\chi : \mathbb{R} \to \mathbb{C}$ such that $\mathbb{E}(e^{iuX_\tau}) = e^{\tau\chi(u)}$, with

$$\chi(u) = -\frac{\sigma^2 u^2}{2} + i\beta u + \int_{|z|<1} (e^{iuz} - 1 - iuz)\nu(dz) + \int_{|z|>1} (e^{iuz} - 1)\nu(dz),$$

for $\sigma \geq 0$, $\beta \in \mathbb{R}$ and a positive measure ν on $\mathbb{R}\backslash\{0\}$ such that $\int_{\mathbb{R}} \min(1, z^2)\nu(dz) < +\infty$. The measure ν is called the Lévy measure of $(X_\tau)_{\tau>0}$.

We assume that the discounted price of the risky asset is a martingale obtained as the exponential of a Lévy process: $e^{-r\tau} S_\tau = S_0 e^{X_\tau}$. The fact that the discounted price is a martingale is equivalent to

$$\int_{|z|>1} e^z \nu(dz) < \infty, \quad \text{and} \quad \beta = -\frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^z - 1 - z 1_{|z|\leq 1})\nu(dz).$$

We also assume that $\int_{|z|>1} e^{2z}\nu(dz) < \infty$, so the discounted price is a square integrable martingale.

In what follows, we assume that the Lévy measure has a density, $\nu(dz) = k(z)dz$, with k possibly singular at $z = 0$. Doing so, we exclude the simplest Lévy processes obtained as the sum of Brownian motions and Poisson processes. This is not a

fundamental restriction in the sense that the methods proposed below could be extended (and even simplified) to calibrate the previously mentioned processes. The restriction is mainly done in order to focus on the difficulties posed by the possible singularities of k at $z = 0$.
We note $\overline{B}$ the integral operator:

$$(\overline{B}v)(S) = \int_{\mathbb{R}} \left(v(Se^z) - v(S) - S(e^z - 1)\frac{\partial}{\partial S}v(S) \right) k(z)dz.$$

Consider an American option with payoff $\overline{P_\circ}$ and maturity t: in [10], Bensoussan and Lions assume $\sigma > 0$ and study the variational inequality stemming from the LCP: $P(t, S) = \overline{P_\circ}(S)$, and for $\tau < t$ and $S > 0$,

$$\frac{\partial P}{\partial \tau}(\tau, S) + \frac{\sigma^2 S^2}{2}\frac{\partial^2 P}{\partial S^2}(\tau, S) + rS\frac{\partial P}{\partial S}(\tau, S) - rP(\tau, S) + (\overline{B}P)(\tau, S) \leq 0, \quad (2.1)$$

$$P(\tau, S) \geq \overline{P_\circ}(S), \quad (2.2)$$

$$\left(\begin{array}{c} \frac{\partial P}{\partial \tau}(\tau, S) + \frac{\sigma^2 S^2}{2}\frac{\partial^2 P}{\partial S^2}(\tau, S) + rS\frac{\partial P}{\partial S}(\tau, S) \\ -rP(\tau, S) + (\overline{B}P)(\tau, S) \end{array} \right) (P(\tau, S) - \overline{P_\circ}(S)) = 0, \quad (2.3)$$

in suitable Sobolev spaces with decaying weights near $+\infty$ and 0, and prove that the price of the American option is $P_\tau = P(\tau, S_\tau)$. Other approaches with viscosity solutions are possible, see [35], especially in the case $\sigma = 0$. One advantage of the variational methods is that they provide stability estimates. For numerical methods for options on Lévy driven assets, see [4, 16, 17, 29, 30, 31].

2.2. The forward linear complementarity problem. As already explained, we aim at finding a forward LCP in the variables maturity/strike; a single solution of this problem will be needed for evaluating the cost function in (1.2).
Hereafter, since the observed prices are those of vanilla American put options, we use the notation

$$P_\circ(x) = (x - S)_+. \quad (2.4)$$

If $\overline{P}_\circ(S) = (x - S)_+$, it can be seen that the solution of (2.1)-(2.3) is of the form

$$P(\tau, S, t, x) = xg(\xi, y), \quad y = S/x \in \mathbb{R}_+, \xi = t - \tau \in (0, t), \quad (2.5)$$

where g is the solution of the complementarity problem independent of x, $g(0, y) = (1 - y)_+$ and for $0 < \xi \leq t$, $y \in \mathbb{R}_+$,

$$-\frac{\partial g}{\partial \xi}(\xi, y) + \frac{\sigma^2 y^2}{2}\frac{\partial^2 g}{\partial y^2}(\xi, y) + ry\frac{\partial g}{\partial y}(\xi, y) - rg(\xi, y) + (\check{B}g)(\xi, y) \leq 0, \quad (2.6)$$

$$g(\xi, y) \geq (1 - y)_+, \quad (2.7)$$

$$\left(\begin{array}{c} -\frac{\partial g}{\partial \xi}(\xi, y) + \frac{\sigma^2 y^2}{2}\frac{\partial^2 g}{\partial y^2}(\xi, y) + ry\frac{\partial g}{\partial y}(\xi, y) \\ -rg(\xi, y) + (\check{B}g)(\xi, y) \end{array} \right) (g(\xi, y) - (1 - y)_+) = 0, \quad (2.8)$$

where $(\check{B}v)(y) = \int_{\mathbb{R}} \left(v(ye^z) - v(y) - y(e^z - 1)\frac{\partial}{\partial y}v(y) \right) k(z)dz$. From this observation and the identities $x\frac{\partial g}{\partial \xi} = -\frac{\partial P}{\partial t}$, $xy\frac{\partial g}{\partial y} = -x\frac{\partial P}{\partial x} + P$, and $xy^2\frac{\partial^2 g}{\partial y^2} = x^2\frac{\partial^2 P}{\partial x^2}$, we deduce that, as a function of t and x, $P(0, S, t, x)$ satisfies the following forward problem: $P(t = 0) = P_\circ$ and for $t \in (0, T]$, and $x > 0$,

$$\left(\frac{\partial P}{\partial t} - \frac{\sigma^2 x^2}{2}\frac{\partial^2 P}{\partial x^2} + rx\frac{\partial P}{\partial x} + BP \right) \geq 0, \quad (2.9)$$

$$P(t, x) \geq P_\circ(x), \quad (2.10)$$

$$\left(\frac{\partial P}{\partial t} - \frac{\sigma^2 x^2}{2}\frac{\partial^2 P}{\partial x^2} + rx\frac{\partial P}{\partial x} + BP \right) (P - P_\circ) = 0, \quad (2.11)$$

where the integral operator B is defined by

$$(Bu)(x) = -\int_{\mathbb{R}} k(z)\left(x(e^z-1)\frac{\partial u}{\partial x}(x) + e^z(u(xe^{-z}) - u(x))\right)dz. \tag{2.12}$$

Note that the arguments yielding (2.9)-(2.11) are much easier than those used for getting Dupire's equation, see [4, 18], because (2.5) does not hold with local volatility. Problem (2.9)-(2.11) can also be obtained by probabilistic arguments. Note also that finding a forward LCP in the variables t and x is not possible in the case of American options with local volatility, because the arguments in [4, 18] do not apply to nonlinear problems. This explains why, in [2, 5], the evaluation of the least square cost functional necessitates the solution of $\#I$ LCP instead of one here. In this respect, we may say that with American options, the calibration of Lévy processes is easier than the calibration of local volatility.

2.3. Choice of the Lévy process. We have already discussed our choice to take $\nu(dz) = k(z)dz$, with

$$\max\left(\int_{\mathbb{R}} \min(1, z^2)k(z)dz, \int_1^{+\infty} e^{2z}k(z)dz\right) < \infty. \tag{2.13}$$

We need to make further restrictions on the Lévy process for several reasons

1. in practice, we need to specify a class of Lévy densities k in order to define the inverse problem.
2. the analysis below will need problem (2.16-2.19) to be parabolic. This implies restrictions on the pair (σ, k).

As it will appear in section 3.2 below, the restrictions in order to have a parabolic problem are

1. either $\sigma > 0$ and k satisfies (2.13).
2. or $\sigma = 0$ and k satisfies (2.13) and is sufficiently singular near $z = 0$; The result in § 3.2 will imply that choosing $k(z) \sim |z|^{-1-2\alpha}$ with $1/2 < \alpha < 1$ yields a parabolic problem. For keeping the length of this article reasonable, this case will be discussed elsewhere.

Assumption 1. *For this reason, we assume that k is of the form*

$$k(z) = \psi(z)|z|^{-(1+2\alpha)}, \tag{2.14}$$

where ψ is a nonnegative function in $L^\infty(\mathbb{R})$ such that $\psi(z) \geq \underline{\psi} > 0$ in a fixed neighborhood of $z = 0$, and α is s.t. $-1/2 \leq \alpha < 1$. We assume furthermore that (2.13) is satisfied. For practical purpose, one can impose further restrictions on ψ, for example let ψ belong to a finite dimensional function space, but this needs not be discussed at this stage.

Assumption 1 holds for models of jump-diffusion type, for example Merton model ($\sigma > 0$ and the jumps in the log-price have a Gaussian distribution) or some Kou models ($\sigma > 0$ and the distribution of jumps is an asymmetric exponential with a fast enough decay at infinity), see [13], page 111. Indeed these models can be obtained by taking $\alpha = -1/2$ and choosing ψ properly. Assumption 1 also holds for some variance gamma processes ($\sigma > 0,\ \alpha = 0$) and normal inverse Gaussian processes ($\sigma > 0,\ \alpha = 1/2$), see [13], page 117, with a fast enough decay of the jump density at infinity. It also holds for some tempered stable processes, see [13], page 119, or some parabolic CGMY models discussed by Carr et al [12]. These last two models usually take $\sigma = 0$. Allowing $\sigma > 0$ in the analysis can be seen as a step toward $\sigma = 0$.

Remark 1. *The assumption $\psi(z) \geq \underline{\psi} > 0$ near 0 avoids ambiguities in the definition of the singularity of k at $z = 0$. It is a bit restrictive since for example a logarithmic singularity of k at $z = 0$ is ruled out. However, this assumption is unessential and most of the results below hold without it.*

2.4. Change of unknown function in the forward problem. In order to have a datum with a compact support in x, it is helpful to change the unknown function: we set

$$u_\circ(x) = (S - x)_+; \qquad u(t,x) = P(t,x) - x + S. \tag{2.15}$$

The function u satisfies: for $t \in (0,T]$, and $x > 0$,

$$\frac{\partial u}{\partial t} - \frac{\sigma^2 x^2}{2}\frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} + Bu \geq -rx, \tag{2.16}$$

$$u(t,x) \geq u_\circ(x), \tag{2.17}$$

$$\left(\frac{\partial u}{\partial t} - \frac{\sigma^2 x^2}{2}\frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} + Bu + rx\right)(u - u_\circ) = 0. \tag{2.18}$$

The initial condition for u is

$$u(t=0,x) = u_\circ(x), \; x > 0. \tag{2.19}$$

For writing the variational inequalities stemming from (2.16)-(2.19), we need to introduce suitable weighted Sobolev spaces. In particular, fractional order weighted Sobolev spaces will be useful for studying the nonlocal part of the operator.

3. Preliminary results.

3.1. Functional setting.

3.1.1. Sobolev spaces on $\mathbb{R}$. For a real number s, let the Sobolev space $H^s(\mathbb{R})$ be defined as follows: the distribution w defined on $\mathbb{R}$ belongs to $H^s(\mathbb{R})$ if and only if its Fourier transform $\widehat{w}$ satisfies $\int_{\mathbb{R}}(1+\xi^2)^s|\widehat{w}(\xi)|^2 d\xi < +\infty$. The spaces $H^s(\mathbb{R})$ are Hilbert spaces, with the inner product and norm:

$$(w_1,w_2)_{H^s(\mathbb{R})} = \int_{\mathbb{R}} (1+\xi^2)^s \widehat{w_1}(\xi)\overline{\widehat{w_2}}(\xi) d\xi, \qquad \|w\|_{H^s(\mathbb{R})} = \sqrt{(w,w)_{H^s(\mathbb{R})}}.$$

For two real numbers $s_1, s_2, s_1 \leq s_2$, $H^{s_2}(\mathbb{R}) \subset H^{s_1}(\mathbb{R})$ with a continuous injection. It can be seen that $H^0(\mathbb{R}) = L^2(\mathbb{R})$ and that if s is a positive integer, $H^s(\mathbb{R})$ is the space of all the functions whose derivatives up to order s are square integrable. If s is a nonnegative integer, the norm $\|.\|_{H^s(\mathbb{R})}$ is equivalent to the norm $v \mapsto \sqrt{\sum_{\ell=0}^s \|\frac{d^\ell v}{dy^\ell}\|^2_{L^2(\mathbb{R})}}$. If $s > 0$ is not an integer, the norm $\|.\|_{H^s(\mathbb{R})}$ is equivalent to

$$v \mapsto \sqrt{\sum_{\ell=0}^m \|\frac{d^\ell v}{dy^\ell}\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}}\int_{\mathbb{R}} |y-z|^{2(m-s)-1}\left(\frac{d^m v}{dy^m}(y) - \frac{d^m v}{dy^m}(z)\right)^2 dydz}, \tag{3.1}$$

where m is the integer part of s. For $s \geq 0$, the space $\mathcal{D}(\mathbb{R})$ is dense in $H^s(\mathbb{R})$.
It is well known (see [27, 6]) that if $0 < s < 1$, then $H^s(\mathbb{R})$ can be obtained by real or complex interpolation between the spaces $H^1(\mathbb{R})$ and $L^2(\mathbb{R})$ (the parameter for the real interpolation is $\nu = 1/2 - s$, see [6] page 204), and that the norm obtained by the interpolation process is equivalent to the one defined in (3.1).
For $s \geq 0$, $H^{-s}(\mathbb{R})$ is the dual of $H^s(\mathbb{R})$, and for $s > 0$, the norm $\|.\|_{H^{-s}(\mathbb{R})}$ is equivalent to the norm $v \mapsto \sup_{w \in H^s(\mathbb{R}), w \neq 0} \frac{|\langle v,w\rangle|}{\|w\|_{H^s(\mathbb{R})}}$. If s is a nonnegative integer, we define the semi-norm $|v|_{H^s(\mathbb{R})} = \sqrt{\sum_{\ell=1}^s \|\frac{d^\ell v}{dy^\ell}\|^2_{L^2(\mathbb{R})}}$. If $s > 0$ is not an integer, we define $|v|_{H^s(\mathbb{R})}$ by $|v|^2_{H^s(\mathbb{R})} = \sum_{\ell=1}^m \|\frac{d^\ell v}{dy^\ell}\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\left(\frac{d^m v}{dy^m}(y) - \frac{d^m v}{dy^m}(z)\right)^2}{|y-z|^{1+2s}}$, where m is the integer part of s.

3.1.2. Some weighted Sobolev spaces on $\mathbb{R}_+$. Let V^1 be the weighted Sobolev space

$$V^1 = \left\{ v \in L^2(\mathbb{R}_+), x\frac{\partial v}{\partial x} \in L^2(\mathbb{R}_+) \right\},$$

which is a Hilbert space with the norm $\|v\|_{V^1} = \sqrt{\|v\|^2_{L^2(\mathbb{R}_+)} + \|x\frac{\partial v}{\partial x}\|^2_{L^2(\mathbb{R}_+)}}$. It is proved in [4] that $\mathcal{D}(\mathbb{R}_+)$ is a dense subspace of V^1, and that the following Poincaré inequality is true: for all $v \in V^1$,

$$\|v\|_{L^2(\mathbb{R}_+)} \leq 2\|x\frac{dv}{dx}\|_{L^2(\mathbb{R}_+)}. \tag{3.2}$$

Therefore, the semi-norm $|.|_{V^1}$: $|v|_{V^1} = \|x\frac{dv}{dx}\|_{L^2(\mathbb{R}_+)}$ is a norm equivalent to $\|.\|_{V^1}$.
For a function v defined on $\mathbb{R}_+$, call $\tilde{v}$ the function defined on $\mathbb{R}$ by

$$\tilde{v}(y) = v(\exp(y))\exp(y/2). \tag{3.3}$$

By using the change of variable $y = \log(x)$, it can be seen that the mapping $v \mapsto \tilde{v}$ is a topological isomorphism from $L^2(\mathbb{R}_+)$ onto $L^2(\mathbb{R})$, and from V^1 onto $H^1(\mathbb{R})$. This leads to defining the space V^s, for $s \in \mathbb{R}$, by

$$V^s = \{v \ : \ \tilde{v} \in H^s(\mathbb{R})\},$$

which is a Hilbert space with the norm $\|v\|_{V^s} = \|\tilde{v}\|_{H^s(\mathbb{R})}$. Using the interpolation theorem given e.g. in [6] Theorem 7.17, one can prove that if $0 < s < 1$, then V^s can be obtained by real interpolation between the spaces V^1 and $L^2(\mathbb{R}_+)$ (the parameter for the real interpolation is $\nu = 1/2 - s$), and that the norm obtained by the interpolation process is equivalent to the one defined above.
For $s > 0$, the space V^{-s} is the topological dual of V^s.
For $s > 0$, we introduce the semi-norm $|v|_{V^s} = |\tilde{v}|_{H^s(\mathbb{R})}$.

LEMMA 3.1. *Let s be a real number such that $1/2 < s \leq 1$. Then for all $v \in V^s$, v is continuous on $(0, +\infty)$ and there exists a constant $C > 0$ such that*

$$\sqrt{x}|v(x)| \leq C\|v\|_{V^s}, \quad \forall x \in [1, +\infty). \tag{3.4}$$

Proof. From the Sobolev continuous imbedding $H^s(\mathbb{R}) \subset L^\infty(\mathbb{R}) \cap C^0(\mathbb{R})$ for $s > 1/2$, we see that $V^s \subset \mathcal{C}^0((0, +\infty))$ and there exists a constant C such that $|v(x)| = |\tilde{v}(\log(x))|/\sqrt{x} \leq C\|\tilde{v}\|_{H^s(\mathbb{R})}/\sqrt{x} = C\|v\|_{V^s}/\sqrt{x}$, $\forall v \in V^s$, $\forall x \geq 1$. ∎
For a continuous and nonnegative function ϕ defined on $\mathbb{R}$, and a measurable function v on $\mathbb{R}_+$, consider

$$|v|^2_{\phi,s} = \int_{\mathbb{R}_+} dx \int_{\mathbb{R}} \frac{\phi(z)}{|z|^{1+2s}} \left(v(xe^{-z}) - v(x)\right)^2 dz, \text{ and } \|v\|_{\phi,s} = \sqrt{|v|^2_{\phi,s} + \|v\|^2_{L^2(\mathbb{R}_+)}}.$$

LEMMA 3.2. *Let ϕ be a continuous and nonnegative function defined on $\mathbb{R}$. If $\phi(0) > 0$ and if the function $z \mapsto \phi(z)\max(e^z, 1)$ is bounded, then for any $s \in (0, 1)$, $\|.\|_{\phi,s}$ is a norm on V^s equivalent to the norm $\|.\|_{V^s}$.*

Proof. For the reader's ease, the proof is postponed to § x7. ∎

Remark 2. *Lemma 3.2 remains true if ϕ is a function in $L^\infty(\mathbb{R})$ and if for a given positive constant $\underline{\phi}$, $\phi \geq \underline{\phi} > 0$ a.e. in a neighborhood of 0.*

Remark 3. *If the assumption $\phi(0) > 0$ is not satisfied, then the conclusion of Lemma 3.2 becomes: $\exists C > 0$ such that $|u|_{\phi,s} \leq C\|u\|_{V^s}$, $\forall u \in V^s$.*

3.2. The integro-differential operator.

3.2.1. The integral operator. We study the operator B defined in (2.12).

LEMMA 3.3. *Let (α, ψ) satisfy Assumption 1. For each $s \in \mathbb{R}$,*
if $\alpha > 1/2$, then the operator B is continuous from V^s to $V^{s-2\alpha}$,
if $\alpha < 1/2$, then the operator B is continuous from V^s to V^{s-1},
if $\alpha = 1/2$, then the operator B is continuous from V^s to $V^{s-1-\epsilon}$, for any $\epsilon > 0$.

Proof. See § 7. ☐

COROLLARY 3.4. *If (α, ψ) satisfy Assumption 1 and if $1/2 < \alpha < 1$, then the operator B is continuous from V^α to $V^{-\alpha}$.*

LEMMA 3.5. *If (α, ψ) satisfy Assumption 1 and $1/2 < \alpha < 1$, then $\forall v, w \in V^\alpha$,*

$$\langle Bu, v\rangle + \langle Bv, u\rangle = \begin{cases} \int_{\mathbb{R}_+}\int_{\mathbb{R}} k(z)e^z(u(x) - u(xe^{-z}))(v(x) - v(xe^{-z}))dxdz \\ + \left(\int_{\mathbb{R}} k(z)(2e^z - e^{2z} - 1)dz\right) \int_{\mathbb{R}_+} u(x)v(x)dx \end{cases} \tag{3.5}$$

where $\langle\ ,\ \rangle$ stands for the duality pairing between $V^{-\alpha}$ and V^α.
If $-1/2 \le \alpha \le 1/2$, then, (3.5) is true for $u, v \in V^s$, $s > 1/2$, defining $\langle\ ,\ \rangle$ as the duality pairing between V^{-s} and V^s.

Proof. See § 7. ☐

Remark 4. *If (α, ψ) satisfy Assumption 1, the operator B^T defined by*

$$B^T u(x) = \int_{\mathbb{R}} k(z)\left(x(e^z - 1)\frac{\partial u}{\partial x}(x) - e^{2z}u(xe^z) + (2e^z - 1)u(x)\right)dz \tag{3.6}$$

is a continuous operator from V^s to $V^{s-2\alpha}$, if $\alpha > 1/2$,
is a continuous operator from V^s to V^{s-1}, if $\alpha < 1/2$,
is a continuous operator from V^s to $V^{s-1-\epsilon}$, for any $\epsilon > 0$, if $\alpha = 1/2$.
If $\alpha > 1/2$, then $\forall u, v \in V^\alpha$, $\langle B^T u, v\rangle = \langle Bv, u\rangle$. This identity holds for all $u, v \in V^s$ with $s > 1/2$ if $\alpha \le 1/2$.

LEMMA 3.6. *If (α, ψ) satisfy Assumption 1 and if*
- *either $\alpha < 1/2$,*
- *or ψ is continuous near 0 and there exists a bounded function $\omega : \mathbb{R} \to \mathbb{R}$ and two positive numbers ζ and C such that $\psi(z)e^{3/2z} - \psi(0)e^{-3/2z} = z\omega(z)$, with $|\omega(z)| \le C|z|e^{-\zeta|z|}$, for all $z \in \mathbb{R}$,*

then for any $s \in \mathbb{R}$, the operator $B - B^T$ is continuous from V^s to V^{s-1}.

Proof. See § 7. ☐

PROPOSITION 3.7 (Gårding inequality). *Let (α, ψ) satisfy Assumption 1. If $1/2 < \alpha < 1$, there exist two constants $\underline{C} > 0$ and $\lambda \ge 0$ such that, $\forall v \in V^\alpha$,*

$$\langle Bv, v\rangle \ge \underline{C}|v|^2_{V^\alpha} - \lambda\|v\|^2_{L^2(\mathbb{R}_+)}. \tag{3.7}$$

If $\alpha \le 1/2$, then (3.7) holds for any $v \in V^s$, $s > 1/2$ ($\langle,\rangle$ standing for the duality pairing between V^{-s} and V^s), with $\underline{C} = 0$ if $\alpha < 0$.

Proof. If $0 < \alpha < 1$, the function $\phi : z \mapsto e^z\psi(z)$ satisfies the assumptions of Remark 2. Therefore, $u \mapsto \sqrt{\|u\|^2_{L^2(\mathbb{R}_+)} + \int_{\mathbb{R}_+}\int_{\mathbb{R}} k(z)e^z(u(x) - u(xe^{-z}))^2 dxdz}$ is a norm on V^α equivalent to the norm $\|.\|_{V^\alpha}$. From this and (3.5), we deduce (3.7). ☐

Consider the two situations
- $1/2 < \alpha < 1$, ψ and $u \in V^\alpha$: it can be shown (using the interpolation theorem 7.17 in [6]) that the functions u_+ and u_- belong to V^α;
- $\alpha \le 1/2$ and $u \in V^s$, $s > 1/2$.

In both cases, $\int_{\mathbb{R}_+}\int_{\mathbb{R}} k(z)e^z u_-(xe^{-z})u_+(x)dxdz$ is well defined because

$$\begin{aligned}\int_{\mathbb{R}_+}\int_{\mathbb{R}} k(z)e^z u_-(xe^{-z})u_+(x)dxdz &= \int_{\mathbb{R}_+}\int_{\mathbb{R}} k(z)e^z(u_-(xe^{-z}) - u_-(x))u_+(x)dxdz \\ &\le C\|u_+\|_{L^2(\mathbb{R}_+)}\sqrt{\int_{\mathbb{R}_+}\int_{\mathbb{R}} k(z)e^z(u_-(xe^{-z}) - u_-(x))^2 dzdx},\end{aligned}$$

and is nonnegative. Therefore,

$$\begin{aligned}\langle Bu, u_+\rangle &= \langle Bu_+, u_+\rangle - \int_{\mathbb{R}_+}\int_{\mathbb{R}} k(z)e^z(u(xe^{-z}) - u_+(xe^{-z}))u_+(x)dxdz \\ &= \langle Bu_+, u_+\rangle + \int_{\mathbb{R}_+}\int_{\mathbb{R}} k(z)e^z u_-(xe^{-z})u_+(x)dxdz \geq \langle Bu_+, u_+\rangle.\end{aligned}$$

We have proved the

LEMMA 3.8. *If (α, ψ) satisfy Assumption 1 then there exist two constants $\underline{C} > 0$ and $\lambda \geq 0$ such that, for all $u \in V^\alpha$ if $\alpha > 1/2$ or for all $u \in V^s$ $s > 1/2$ if $\alpha \leq 1/2$,*

$$\langle Bu, u_+\rangle \geq \underline{C}|u_+|^2_{V^\alpha} - \lambda\|u_+\|^2_{L^2(\mathbb{R}_+)}, \tag{3.8}$$

with $\underline{C} = 0$ if $\alpha < 0$.

3.2.2. The integro-differential operator. With B defined in (2.12), we introduce the integro-differential operator A:

$$Av = -\frac{\sigma^2 x^2}{2}\frac{\partial^2 v}{\partial x^2} + rx\frac{\partial v}{\partial x} + Bv, \tag{3.9}$$

where σ and r are nonnegative real numbers. In this work, we limit ourselves to the case $\sigma > 0$. The case $\sigma = 0$, $\alpha > 1/2$ requires working in the fractional Sobolev spaces described above and will be treated in [1]. Since the space V^1 will play a special role, we use the shorter notation $V = V^1$.

If $\sigma > 0$, and if (α, ψ) satisfy Assumption 1, then

- A is a continuous operator from V to V^{-1},
- we have the Gårding inequalities: there exist $\underline{c} > 0$ and $\lambda \geq 0$ such that

$$\langle Av, v\rangle \geq \underline{c}|v|^2_V - \lambda\|v\|^2_{L^2(\mathbb{R}_+)}, \quad \forall v \in V, \tag{3.10}$$

$$\langle Av, v_+\rangle \geq \underline{c}|v_+|^2_V - \lambda\|v_+\|^2_{L^2(\mathbb{R}_+)}, \quad \forall v \in V. \tag{3.11}$$

- The operator $A + \lambda I$ is one to one and continuous from V^2 onto $L^2(\mathbb{R}_+)$, with a continuous inverse.

Remark 5. *The assumption that $\psi > 0$ near $z = 0$ is not necessary for A to have the above properties. Its role is to allow a clear identification of the kernel's singularity at $z = 0$.*

3.3. The variational inequalities. We are ready to write the variational inequalities corresponding to the LCP (2.16)-(2.19).

We introduce the closed subspace of V:

$$K = \{v \in V,\ v(x) \geq u_\circ(x) \text{ in } \mathbb{R}_+\}. \tag{3.12}$$

The variational problem will consist of looking for $u \in L^2(0, T; V) \cap C^0([0, T]; L^2(\mathbb{R}_+))$, with $\frac{\partial u}{\partial t} \in L^2((0, T) \times \mathbb{R}_+)$, such that

1. there exists a constant $X_T > S$ such that $u(t, x) = 0$, $\forall t \in [0, T]$, $\forall x \geq X_T$.
2. $u(t) \in K$ for almost every $t \in (0, T)$.
3. For almost every $t \in (0, T)$, for any $v \in K$ with bounded support,

$$\left\langle \frac{\partial u}{\partial t} + Au + rx, v - u\right\rangle \geq 0, \tag{3.13}$$

   where $\langle , \rangle$ stands for the duality pairing between V' (the dual of V) and V.
4. $u(t = 0) = u_\circ$.

Hereafter, this problem will be referred to as (VIP). The goal of section 4 below is to prove that (VIP) has a unique solution and to study its properties.

4. Analysis of the variational inequalities.

4.1. Orientation. Hereafter, we assume that $\sigma > 0$.
Problem (2.16)-(2.19) is posed in an unbounded domain. This is a technical difficulty in order to use variational methods, and we first have to replace this problem by a similar one posed in a bounded domain. Therefore, the program is to

1. approximate (2.16)-(2.19) by a similar problem posed in $[0, T] \times [0, X]$, for some given positive parameter $X > S$, and write the related variational problem, which will be called (VIP$_X$) below;
2. solve first a penalized version of (VIP$_X$) by introducing a semilinear monotone operator. Pass to the limit as the penalty parameter tends to zero;
3. prove that the free boundary of (VIP$_X$) stays in a bounded domain as X tends to infinity: this will show that for X large enough a solution of (VIP$_X$) is actually a solution of (VIP);
4. obtain estimates for the solution of (VIP) independent of the parameters (σ, α, ψ), when these parameters vary in a suitably defined set.

4.2. Approximation of (VIP) in a bounded domain. Let X be a positive number greater than S. Hereafter, for a function $v \in L^2((0, X))$ we call $\mathcal{E}_X(v)$ the function in $L^2(\mathbb{R}_+)$ obtained by extending v by 0 outside $(0, X)$. We introduce the Sobolev spaces $W^1_X = \{v \in L^2((0, X)), x\frac{\partial v}{\partial x} \in L^2((0, X))\}$ and $W^2_X = \{v \in W^1_X, x^2\frac{\partial^2 v}{\partial x^2} \in L^2((0, X))\}$. For β, $0 < \beta < 1$, W^β_X is the space obtained by real interpolation between W^1_X and $L^2(0, X)$ with parameter $\nu = 1/2 - \beta$, (see [6] page 204,[27]), and $W^{1+\beta}_X = \{v \in W^1_X, x\frac{\partial v}{\partial x} \in W^\beta_X\}$.
For β, $0 \le \beta < 3/2$, we introduce $V^\beta_X = \{v \in L^2(0, X), \mathcal{E}(v) \in V^\beta\}$, endowed with the norm $\|v\|_{V^\beta_X} = \|\mathcal{E}_X(v)\|_{V^\beta}$. Note that for β, $0 \le \beta < 1/2$, $V^\beta_X = W^\beta_X$. Let $V^{-\beta}_X$ be the dual of V^β_X. Thanks to Lemma 3.1, we know that for $\beta > 1/2$, a function $v \in V^\beta_X$ is continuous in $[0, X]$ and vanishes at X.
Since the space V^1_X will often be used, we introduce the special notations

$$V_X = \{v \in L^2(0, X), \mathcal{E}_X(v) \in V\}, \tag{4.1}$$

and $\|v\|_{V_X} = \|\mathcal{E}_X(v)\|_V$. We define the operators A_X and B_X, $V_X \to V'_X$,

$$\langle A_X v, w\rangle = \langle A\mathcal{E}_X(v), \mathcal{E}_X(w)\rangle \quad \text{and} \quad \langle B_X v, w\rangle = \langle B\mathcal{E}_X(v), \mathcal{E}_X(w)\rangle. \tag{4.2}$$

A Gårding inequality for A_X is deduced from (3.10), with constants independent of X. We define

$$D_X = \{v \in V_X : \; A_X v \in L^2((0, X))\}. \tag{4.3}$$

It follows from the Gårding inequality that (A_X, D_X) is the infinitesimal generator of an analytic semi-group [34]. Proposition 4.1 below contains information on D_X:

PROPOSITION 4.1. *If $v \in D_X$, then for any number $X' < X$, $v|_{(0,X')} \in W^2_{X'}$. For α, $0 < \alpha < 3/4$, $D_X = W^2_X \cap V_X$. For α, $3/4 \le \alpha < 1$, there exists $\epsilon > 0$ such that $D_X \subset W^{3/2+\epsilon}_X \cap V_X$. In any case, if $v \in D_X$, then $\frac{\partial v}{\partial x} \in \mathcal{C}^0((0, X])$.*

Proof. See § 8. ∎

Remark 6. *It can be proved by lengthy calculations that if $v(x) = X - x$, (note that $v \in W^2_X \cap V_X$) then $A_X v$ behaves like $(X - x)^{1-2\alpha}$ near $x = X$, so $A_X v \notin L^2((0, X))$ if $\alpha > 3/4$.*
We introduce

$$K_X = \{v \in V_X, \; v(x) \ge u_\circ(x) \; \text{ in } (0, X)\}. \tag{4.4}$$

We are going to look for $u_X \in L^2(0, T; V_X) \cap C^0([0, T]; L^2((0, X)))$, with $\frac{\partial u_X}{\partial t} \in L^2((0, T) \times (0, X))$, such that

1. $u_X(t) \in K_X$ for almost every $t \in (0,T)$.
2. For almost every $t \in (0,T)$,

$$\left\langle \frac{\partial u_X}{\partial t} + A_X u_X + rx, v - u_X \right\rangle \geq 0, \tag{4.5}$$

for any $v \in K_X$. Here $\langle , \rangle$ stands for the duality pairing between V_X' (the dual of V_X) and V_X.
3. $\mathcal{E}_X(u_X)(t=0) = u_\circ$.

Hereafter, this problem will be referred to as (VIP$_X$). In order to prove that (VIP$_X$) has a unique solution, we follow [25] and introduce first a sequence of monotone problems which can be seen as penalized versions of (4.5): find $u_{X,\epsilon}$ such that

$$\begin{aligned} &\frac{\partial u_{X,\epsilon}}{\partial t} + A_X u_{X,\epsilon} + rx(1 - 1_{\{x>S\}}\mathcal{V}_\epsilon(u_{X,\epsilon})) = 0, \; t \in (0,T], \; 0 < x < X, \\ &u_{X,\epsilon}(t=0,x) = u_\circ(x), \quad 0 < x < X, \\ &u_{X,\epsilon}(t,X) = 0, \quad t \in (0,T], \end{aligned} \tag{4.6}$$

where $\mathcal{V}_\epsilon(u) = \mathcal{V}(u/x\epsilon)$ and $\mathcal{V}$ is a smooth nonincreasing convex function such that

$$\mathcal{V}(0) = 1, \qquad \mathcal{V}(u) = 0 \quad \text{for } u \geq 1, \qquad 0 \geq \mathcal{V}'(u) \geq -2 \quad \text{for } 0 \leq u \leq 1.$$

In what follows, we call $u_X^{(E)}$ and $\underline{u}_X^{(E)}$ the solutions to the linear problems:

$$\begin{array}{rcll} \dfrac{\partial u_X^{(E)}}{\partial t} + A_X u_X^{(E)} = 0, \quad \dfrac{\partial \underline{u}_X^{(E)}}{\partial t} + A_X \underline{u}_X^{(E)} &=& -rx, & t \in (0,T], \; 0 < x < X, \\ u_X^{(E)}(t=0,x) = \underline{u}_X^{(E)}(t=0,x) &=& u_\circ(x) & 0 < x < X, \\ u_X^{(E)}(t,X) = \underline{u}_X^{(E)}(t,X) &=& 0 & t \in (0,T], \end{array}$$

It can be seen that $\underline{u}_X^{(E)}(t,0) = S$, $\forall t \in [0,T]$ and that

$$\underline{u}_X^{(E)}(t,x) > S - x, \quad \forall (t,x) \in (0,T] \times (0,X]. \tag{4.7}$$

Let $u^{(E)}$ be the solution of the linear problem:

$$\frac{\partial u^{(E)}}{\partial t} + A u^{(E)} = 0, \quad t \in (0,T], \; x > 0, \qquad u^{(E)}(t=0,x) = u_\circ(x), \quad x > 0.$$

The function $u^{(E)}$ is smooth near $x = 0$ and $\frac{\partial u^{(E)}}{\partial x}(t,0) = -1$, $\forall t \geq 0$.

THEOREM 4.2. *If (α, ψ) satisfy Assumption 1 and if $\sigma > 0$, then (4.6) has a unique weak solution $u_{X,\epsilon} \in L^2(0,T;V_X) \cap C^0([0,T];L^2(0,X))$. It satisfies*

$$\underline{u}_X^{(E)} \leq u_{X,\epsilon} \leq u_X^{(E)} \leq u^{(E)}. \tag{4.8}$$

The function $u_{X,\epsilon}$ belongs to $C^0([0,T];K_X) \cap L^2(0,T;D_X)$ and is continuous and nondecreasing w.r.t. t. For two positive numbers $\epsilon' < \epsilon$, we have

$$u_{X,\epsilon'} \leq u_{X,\epsilon} \leq u_{X,\epsilon'} + \epsilon. \tag{4.9}$$

The quantities $\|u_{X,\epsilon}\|_{L^\infty(0,T;V_X)}$, $\|u_{X,\epsilon}\|_{L^2(0,T;D_X)}$, $\|\frac{\partial u_{X,\epsilon}}{\partial t}\|_{L^2((0,T)\times(0,X))}$ are bounded independently of ϵ. The quantities $\|u_{X,\epsilon}\|_{L^\infty(0,T;L^2(0,X))}$ and $\|u_{X,\epsilon}\|_{L^2(0,T;V_X)}$ are bounded independently of X.

Proof. See §8. ∎

THEOREM 4.3. *The function $x\frac{\partial u_{X,\epsilon}}{\partial x}$ is the sum of $\tilde{z}_{X,\epsilon} \in \mathcal{C}^0([0,T];L^2(0,X))$ and of $\hat{z}_{X,\epsilon} \in L^2(0,T;V_X)$ such that $\tilde{z}_{X,\epsilon} \leq 0$ and $\lim_{\epsilon\to 0} \|\hat{z}_{X,\epsilon}\|_{L^2(0,T;V_X)} = 0$. Finally, for two numbers X and X' such that $S < X < X'$, for any $\epsilon > 0$,*

$$\mathcal{E}_X(u_{X,\epsilon}) \leq \mathcal{E}_{X'}(u_{X',\epsilon}). \tag{4.10}$$

Proof. See §8. ∎

THEOREM 4.4. *If (α, ψ) satisfy Assumption 1 and if $\sigma > 0$, (VIP$_X$) has a unique solution $u_X \in C^0([0,T]; K_X) \cap L^2(0,T; D_X)$, with $\frac{\partial u_X}{\partial t} \in L^2((0,T)\times(0,X))$. The function u_X is continuous in $[0,T]\times[0,X]$, with $u_X(t,0) = S, \forall t \in [0,T]$, $\underline{u}_X^{(E)} \le u_X \le u_X^{(E)} \le u^{(E)}$, and $u_X(t,x) > u_\circ(x)$ for $0 < t \le T$ and $0 < x \le S$.*
The function u_X is nondecreasing with respect to t, and nonincreasing with respect to x. The quantities $\|\mathcal{E}_X(u_X)\|_{L^\infty(0,T;L^2(\mathbb{R}_+))}$ and $\|\mathcal{E}_X(u_X)\|_{L^2(0,T;V)}$ are bounded independently of X.
For $\epsilon > 0$, we have the bounds

$$u_X \le u_{X,\epsilon} \le u_X + \epsilon, \tag{4.11}$$

and the sequence $u_{X,\epsilon}$ converges to u_X uniformly as $\epsilon \to 0$.
For two numbers X and X' such that $S < X < X'$, $\mathcal{E}_X(u_X) \le \mathcal{E}_{X'}(u_{X'})$.

Proof. The proof mainly consists of passing to the limit in (4.6) as $\epsilon \to 0$. It uses the Minty trick, see [25]. We skip it since it is rather classical. ∎

LEMMA 4.5. *If (α, ψ) satisfy Assumption 1 and if $\sigma > 0$, there exists a nondecreasing function $\gamma_X : (0,T] \to (S,X]$, such that the set $\{(t,x) : u_X(t,x) = u_\circ(x)\}$ coincides with the set $\{(t,x) : x \ge \gamma_X(t)\}$. Calling*

$$\mu_X = \frac{\partial u_X}{\partial t} + A_X u_X + rx, \tag{4.12}$$

we have a.e.

$$0 \le \mu_X \le rx 1_{\{u_X=0\}} = rx 1_{\{x \ge \gamma_X(t)\}}. \tag{4.13}$$

Proof. We know that for all $t \in [0,T]$, $u_X(t,X) = u_\circ(X) = 0$. Thus, at each time t, the set where $u_X(t,x)$ coincides with $u_\circ$ is nonempty. It is closed since u_X and $u_\circ$ are continuous. We also know that $u_X(t,x) > u_\circ(x)$ for $t > 0$ and $x \le S$; thus, $\{x > 0 \text{ s.t. } u_X(t,x) = u_\circ(x)\} \subset (S,X]$ for $t > 0$. On the other hand, for all $t \in (0,T]$, the function $u_X(t)$ is nonincreasing with respect to x, so $\{x > 0 \text{ s.t. } u_X(t,x) = u_\circ(x)\}$ is an interval $[\gamma_X(t), X]$, with $\gamma_X(t) > S$. Since u_X is nondecreasing with respect to t, the function γ_X is nondecreasing.
With $\mu_X \in L^2((0,T)\times\mathbb{R}_+)$ given by (4.12), we have $\mu_X = 0$ a.e. in the open region where $u_X > 0$. Now, μ_X is the weak limit of $rx1_{x>S}\mathcal{V}_\epsilon(u_{X,\epsilon})$ in $L^2((0,T)\times(0,X))$. From (4.11), we deduce that $rx1_{x>S}\mathcal{V}_\epsilon(u_{X,\epsilon}) \le rx1_{x>S}\mathcal{V}_\epsilon(u_X)$, and $1_{x>S}\mathcal{V}_\epsilon(u_X)$ converges pointwise to $1_{\{u_X=0\}}$. Therefore, $\mu_X \le rx1_{\{u_X=0\}}$. ∎

PROPOSITION 4.6. *If (α, ψ) satisfy Assumption 1 and if $\sigma > 0$, the function γ_X is nondecreasing and lower semi-continuous. The graph of γ_X has measure 0 (Lebesgue measure in $\mathbb{R}^2$) and*

$$\begin{aligned} \mu_X(t,x) &= 1_{\{u_X(t,x)=0\}}(rx + B_X u_X(t,x)) \\ &= 1_{\{u_X(t,x)=0\}}\left(rx - \int_{\mathbb{R}} k(z)e^z u_X(t, xe^{-z})dz\right) \quad \text{for a.a. } t,x. \end{aligned} \tag{4.14}$$

Proof. We have already seen that γ_X is nondecreasing. The epigraph of γ_X is the set where u_X vanishes. This region is closed since u_X is continuous.
Since γ_X has a left and right limit at each point t, the graph of γ_X has measure 0 (Lebesgue measure in $\mathbb{R}^2$), see Theorem 3.7 in [2] for the proof. As a consequence, the boundary of the coincidence set $\{u_X = 0\}$ has measure 0 (Lebesgue measure in $\mathbb{R}^2$). From this and since the identity $\mu_X(t,x) = rx - \int_{\mathbb{R}} k(z)e^z u_X(t, xe^{-z})dz$ is true in the set $\{x > \gamma_X(t)\}$, we obtain (4.14). ∎

Remark 7. *We will not try to obtain further regularity results on γ_X. Yet, this is certainly an interesting topic on which little seems to be known.*
Let T_X be defined by

$$T_X = \sup\{t,\ 0 < t \leq T,\ \gamma_X(t) < X\}. \tag{4.15}$$

Since γ_X is nondecreasing, we know that if $T_X < T$, then $\forall t \in [T_X, T]$, $\gamma_X(t) = X$. Note that $\mathcal{E}_X(u_X)$ is a solution of (2.16)-(2.19) in $(0, T_X) \times \mathbb{R}_+$, so for all $X' > X$, $\mathcal{E}_X(u_X)$ coincides with $\mathcal{E}_{X'}(u_{X'})$ for $0 < t < T_X$. In particular, this implies that $X \mapsto T_X$ is a nondecreasing function.

Lemma 4.7. *If (α, ψ) satisfy Assumption 1 and if $\sigma > 0$, there exists $X_T > S$ such that for all $X \geq X_T$, $T_X = T$. For $X > X_T$, $u_X \in L^2(0, T; W^2_X)$.*

Proof. The proof is done by contradiction: if X_T does not exist, then $\lim_{X\to\infty} T_X = \underline{T} < T$. We have $\frac{\partial u_X}{\partial t} + A_X u_X = -rx$ in $[\underline{T}, T] \times (0, X)$, for all $X > S$. We choose a smooth and nonnegative function ϕ defined on $\mathbb{R}$ with compact support, and for $y > 0$, we call ϕ_y the function $\phi_y(x) = \phi(x - y)/\sqrt{y}$. Then we take ϕ_y as a test function in (4.12). We have

$$\int_0^X (u_X(T,x) - u_X(\underline{T}, x))\phi_y(x) + \int_{\underline{T}}^T \langle A_X u_X(t), \phi_y\rangle dt = -r\int_{\underline{T}}^T \int_0^X x\phi_y(x)dx.$$

Take $y = X/2$ and let X tend to ∞. From the bounds on u_X, the left hand side in the identity above remains bounded whereas the right hand side tends to infinity. We have obtained the desired contradiction. The last statement of Lemma 4.7 follows easily from the first statement of Proposition 4.1. □

Proposition 4.8. *If (α, ψ) satisfy Assumption 1 and if $\sigma > 0$, the function $\mu_{X,\epsilon} = rx1_{\{x>S\}}\mathcal{V}_\epsilon(u_{X,\epsilon})$, converges to μ_X in $L^p((0,T)\times(0,X))$ for p, $1 \leq p < +\infty$. The sequence $u_{X,\epsilon}$ converges to u_X strongly in $L^2(0,T; D_X)$ and in $L^\infty(0,T; V_X)$.*

Proof. See § 8. □

4.3. The problem (VIP). From Theorem 4.4, Proposition 4.6 and Lemma 4.7, we can pass to the limit as $X \to \infty$:

Theorem 4.9. *If (α, ψ) satisfy Assumption 1 and if $\sigma > 0$, there exists a unique solution of problem (VIP), i.e. a function $u \in C^0([0,T]; K) \cap L^2(0,T; V^2)$, with $\frac{\partial u}{\partial t} \in L^2((0,T) \times \mathbb{R}_+)$, such that $u(t = 0) = u_\circ$,*

$$u(t,x) = 0, \quad \forall t \in [0,T],\ x \geq X_T, \tag{4.16}$$

where X_T is defined in Lemma 4.7, and satisfying the variational inequality (3.13) for all $v \in K$ with bounded support in x. The function u coincides with u_X for $X \geq X_T$. There exists a nondecreasing and lower semi-continuous function γ : $(0,T] \to (S, X_T)$, such that $\forall t \in (0,T)$, $\{x > 0 \text{ s.t. } u(t,x) = u_\circ(x)\} = [\gamma(t), +\infty)$. Calling

$$\mu = \frac{\partial u}{\partial t} + Au + rx, \tag{4.17}$$

we have a.e. $0 \leq \mu \leq rx1_{\{u=0\}} = rx1_{\{x\geq\gamma(t)\}}$.

Proposition 4.10. *The function μ defined in (4.17) is nondecreasing w.r.t. x (i.e. the distribution $\frac{\partial \mu}{\partial x}$ is negative) and nonincreasing w.r.t. t, (i.e. the distribution $\frac{\partial \mu}{\partial t}$ is positive). For any $X > X_T$, the total variation of μ in $(0,T) \times (0,X)$ is bounded by $rX(T + X)$.*

Proof. Consider $X > X_T$. The function μ coincides with μ_X on $(0,T) \times (0,X)$. The monotone character of μ w.r.t. the two variables stems from (4.14) and from the fact that u is nonincreasing w.r.t. x and nondecreasing w.r.t. t.
The same result can be proved by observing that μ_X is the weak limit in $L^2((0,T)\times$

$(0,X))$ of the sequence $rx1_{x>S}\mathcal{V}_\epsilon(u_{X,\epsilon})$ and using the properties of $rx1_{x>S}\mathcal{V}_\epsilon(u_{X,\epsilon})$. The bound on the total variation of μ on $(0,T)\times(0,X)$ comes from the fact that μ is nondecreasing w.r.t. x, nonincreasing w.r.t. t and that $0\le\mu\le rX$ a.e. in $(0,T)\times(0,X)$. ☐

PROPOSITION 4.11. *A.e. in the coincidence set* $\{(t,x):\ u(t,x)=0\}$, $\mu>0$.

Proof. We know from Proposition 4.6 that the boundary of the coincidence set has measure 0 (Lebesgue measure in $\mathbb{R}^2$). Assume that $\mu=0$ in some subset of $x>\gamma(t)$ with positive measure. In view of the monotone behavior of μ, this implies that $\mu=0$ in a rectangle contained in the set $x>\gamma(t)$. From Proposition 4.6, this implies that $\int_{\mathbb{R}}k(z)e^z u(t,xe^{-z})dz=rx$ in this rectangle. Taking the derivative w.r.t. x, we obtain that $\int_{\mathbb{R}}k(z)\frac{\partial u}{\partial x}(t,xe^{-z})dz=r$ in the rectangle. But this is impossible, since $u(t,x)$ is nonincreasing w.r.t. x and non identically 0. ☐

Remark 8. *Proposition 4.11 tells us that there is almost everywhere strict complementarity: the reaction term μ is positive at almost every point where $u=0$.*

4.4. Further bounds. Let us choose some constants $\underline{\sigma}$, $\bar\sigma$, $\underline{\alpha}$, b_1, b_2, $\underline{\psi}$, $\bar\psi$ and $\bar z$ such that $0<\underline{\sigma}\le\bar\sigma$, $0<\underline{\alpha}<1/2$, $b_1>1$, $b_2>1$, $\bar\psi\ge\underline{\psi}>0$ and $\bar z>0$. Let us define the subset $\mathcal{F}$ of $\mathbb{R}_+\times\mathbb{R}\times L^\infty(\mathbb{R})$ by

$$\mathcal{F}=[\underline{\sigma},\bar\sigma]\times[-1/2,1-\underline{\alpha}]\times\left\{\psi:\ \left|\ \begin{array}{l}\|\max(e^{2b_1z},|z|^{b_2},1)\psi\|_{L^\infty(\mathbb{R})}\le\bar\psi;\\ \psi\ge0,\ \psi\ge\underline{\psi}\text{ a.e. in }[-\bar z,\bar z]\end{array}\right.\right\}. \quad (4.18)$$

We can make the three observations:

1. The norm of A as an operator from V to V' is bounded independently of (σ,α,ψ) in $\mathcal{F}$.
2. The constants in (3.10)-(3.11) can be taken independent of (σ,α,ψ) in $\mathcal{F}$.
3. With λ in (3.10) independent of (σ,α,ψ) in $\mathcal{F}$, the operator $A+\lambda I$ is one to one and continuous from V^2 onto $L^2(\mathbb{R}_+)$ and $(A+\lambda I)^{-1}:L^2(\mathbb{R}_+)\mapsto V^2$ is bounded with constants independent of (σ,α,ψ) in $\mathcal{F}$.

By carefully inspecting the proofs of Theorems 4.2, 4.4 and 4.9, we see that

1. The quantities $\|u_{X,\epsilon}\|_{L^\infty(0,T;L^2(0,X))}$ and $\|u_{X,\epsilon}\|_{L^2(0,T;V_X)}$ are bounded independently of (σ,α,ψ) in $\mathcal{F}$.
2. The quantities $\|\mathcal{E}_X(u_X)\|_{L^\infty(0,T;L^2(\mathbb{R}_+))}$ and $\|\mathcal{E}_X(u_X)\|_{L^2(0,T;V)}$ are bounded independently of (σ,α,ψ) in $\mathcal{F}$.
3. The quantities $\|u\|_{L^\infty(0,T;L^2(\mathbb{R}_+))}$ and $\|u\|_{L^2(0,T;V)}$ are bounded independently of (σ,α,ψ) in $\mathcal{F}$.

PROPOSITION 4.12. *The function γ is bounded in $[0,T]$ by some constant $\bar X$ independent of (σ,α,ψ) in $\mathcal{F}$. The quantities $\|u\|_{L^\infty(0,T;V)}$, $\|u\|_{L^2(0,T;V^2)}$ and $\|\frac{\partial}{\partial t}u\|_{L^2((0,T)\times\mathbb{R}_+)}$ are bounded independently of (σ,α,ψ) in $\mathcal{F}$.*

Proof. For a sequence $(\sigma_n,\alpha_n,\psi_n)$ in $\mathcal{F}$, let us call u_n the corresponding solution of problem (VIP), and γ_n the function such that $u_n(t,x)=0\Leftrightarrow x\ge\gamma_n(t)$. Assume that $\lim_{n\to\infty}\gamma_n(T/2)=+\infty$. Then, we can use the same arguments as in the proof of Lemma 4.7 and reach a contradiction. Therefore, $\gamma|_{[0,T/2]}$ is bounded independently of (σ,α,ψ) in $\mathcal{F}$. Since (VIP) can always be solved in $(0,2T)\times\mathbb{R}_+$ instead of $(0,T)\times\mathbb{R}_+$, and since for the solution u, $\|u\|_{L^2(0,2T;V)}+\|\frac{\partial u}{\partial t}\|_{L^2(0,2T;V')}$ is bounded independently of (σ,α,ψ) in $\mathcal{F}$, we can use the same arguments and prove that $\gamma|_{[0,T]}$ is bounded independently of (σ,α,ψ) in $\mathcal{F}$.
Therefore, it is possible to choose $\bar X$ such that, for any $(\sigma,\alpha,\psi)\in\mathcal{F}$, $\gamma<\bar X$, and u coincides with $\mathcal{E}_{\bar X}(u_{\bar X})$ where $u_{\bar X}$ is the solution of (VIP$_{\bar X}$). For $x>\bar X$,

$$\mu(t,x)=rx-\int_{z>\log(x/\bar X)}k(z)e^z u(xe^{-z})dz.$$

Thus, for x large enough, such that, for example, $\log(x/\bar{X}) > 1$,

$$\begin{aligned} 0 \le rx - \mu(t,x) &\le S \int_{z>\log(x/\bar{X})} k(z) e^z dz \le S \int_{z>\log(x/\bar{X})} \psi(z) e^{2z} e^{-z} dz \\ &\le S\bar{\psi} \int_{z>\log(x/\bar{X})} e^{-z} dz = \bar{\psi} S \frac{\bar{X}}{x}. \end{aligned} \tag{4.19}$$

Therefore, $\|\frac{\partial u}{\partial t} + Au\|_{L^2(0,T;L^2(\mathbb{R}_+))}$ is bounded by a constant independent of (σ, α, ψ). This implies that the quantities $\|u\|_{L^\infty(0,T;V)}$, $\|u\|_{L^2(0,T;V^2)}$ and $\|\frac{\partial u}{\partial t}\|_{L^2((0,T)\times\mathbb{R})}$ are bounded independently of $(\sigma, \alpha, \psi) \in \mathcal{F}$. ∎

Remark 9. *It may be possible to impose weaker conditions on ψ, and other choices of $\mathcal{F}$ could be made.*

5. Sensitivity Analysis. Here, we aim at understanding the sensitivity of the solution u of (VIP) and of μ given by (4.17) to the variations of $(\sigma, \alpha, \psi) \in \mathcal{F}$. Let us introduce $\mathcal{B} = \{f : z \mapsto f(z) \max(1, |z|^{b_2}, e^{2b_1 z}) \in L^\infty(\mathbb{R})\}$ endowed with the norm $\|f\|_{\mathcal{B}} = \|f(\cdot) \max(1, |\cdot|^{b_2}, e^{2b_1 \cdot})\|_{L^\infty(\mathbb{R})}$. For $(\sigma, \alpha, \psi) \in \mathcal{F}$, let $u(\sigma, \alpha, \psi)$ be the corresponding solution of (VIP). Accordingly, let $\mu(\sigma, \alpha, \psi)$ be given by (4.17) and $\gamma(\sigma, \alpha, \psi)$ be the function defining the free boundary.

Proposition 5.1. *There exists $C > 0$, such that $(\sigma, \alpha, \psi) \in \mathcal{F}$, $(\tilde{\sigma}, \tilde{\alpha}, \tilde{\psi}) \in \mathcal{F}$,*

$$\|u - \tilde{u}\|_{L^2(0,T;V)} + \|u - \tilde{u}\|_{L^\infty(0,T;L^2(\mathbb{R}_+))} \le C \left(|\sigma - \tilde{\sigma}| + |\alpha - \tilde{\alpha}| + \|\psi - \tilde{\psi}\|_{\mathcal{B}} \right), \tag{5.1}$$

$$\int_0^T \int_{\mathbb{R}} (\mu(\tilde{u} - u_\circ) + \tilde{\mu}(u - u_\circ)) \le C \left(|\sigma - \tilde{\sigma}| + |\alpha - \tilde{\alpha}| + \|\psi - \tilde{\psi}\|_{\mathcal{B}} \right)^2, \tag{5.2}$$

calling $u = u(\sigma, \alpha, \psi)$, $\mu = \mu(\sigma, \alpha, \psi)$, $\tilde{u} = u(\tilde{\sigma}, \tilde{\alpha}, \tilde{\psi})$, $\tilde{\mu} = \mu(\tilde{\sigma}, \tilde{\alpha}, \tilde{\psi})$.

Proof. We skip the proof since its arguments are well known. ∎

Proposition 5.2. *Consider $(\sigma, \alpha, \psi) \in \mathcal{F}$ and let $(\sigma_n, \alpha_n, \psi_n)_{n\in\mathbb{N}}$ be a sequence of coefficients in $\mathcal{F}$ such that $\lim_{n\to\infty} (|\sigma - \sigma_n| + |\alpha - \alpha_n| + \|\psi - \psi_n\|_{\mathcal{B}}) = 0$. Calling $u = u(\sigma, \alpha, \psi)$, $u_n = u(\sigma_n, \alpha_n, \psi_n)$, $\mu = \mu(\sigma, \alpha, \psi)$ and $\mu_n = \mu(\sigma_n, \alpha_n, \psi_n)$,*

$$\lim_{n\to+\infty} \|u_n - u\|_{L^\infty((0,T)\times\mathbb{R}_+)} = 0, \qquad \lim_{n\to+\infty} \|\mu_n - \mu\|_{L^p((0,T)\times\mathbb{R}_+)} = 0, \tag{5.3}$$

for all p, $1 < p < +\infty$, and

$$\|u_n - u\|_{L^\infty(0,T;V^1)} + \|u_n - u\|_{L^2(0,T;V^2)} + \|\frac{\partial(u_n - u)}{\partial t}\|_{L^2((0,T)\times\mathbb{R}_+)} \to 0. \tag{5.4}$$

Proof. From the facts that

- $u(t,x) = u_n(t,x) = 0$ for $x > \bar{X}$ (where $\bar{X}$ is given in Proposition 4.12 and does not depend of $(\sigma, \alpha, \psi) \in \mathcal{F}$),
- $\forall n$, $S - x \le u_n(t,x) \le S$, and $S - x \le u(t,x) \le S$, which implies that $u - u_n$ is arbitrarily small as $x \to 0$ uniformly with respect to n,

it is enough to prove that $\forall \epsilon > 0$,

$$\lim_{n\to+\infty} \|u_n - u\|_{L^\infty((0,T)\times(\epsilon,\bar{X}))} = 0. \tag{5.5}$$

From (5.1), we see that $\lim_{n\to\infty} \|u_n - u\|_{L^\infty(0,T;L^2(\mathbb{R}_+))} = 0$. On the other hand, we know that $\|u_n - u\|_{L^\infty(0,T;V)}$ is bounded independently of n. These two observations imply (5.5), and the first part of (5.3) is proved.

Let us prove the second part of (5.3): from the fact that $\mu_n - rx$ is bounded $L^2((0,T)\times\mathbb{R}_+)$, one can extract a subsequence converging weakly in $L^2((0,T)\times\mathbb{R}_+)$. The limit is nothing else but $\mu - rx$, and the whole sequence $\mu_n - rx$ converges to

$\mu - rx$ weakly in $L^2((0,T)\times\mathbb{R}_+)$. Thanks to (4.19) with $\bar{X}$ independent of $(\sigma,\alpha,\psi)\in\mathcal{F}$, it is enough to prove that μ_n strongly converges to μ in $L^p((0,T)\times(0,X))$, for any $X > S$, and $1 < p < +\infty$. But, from Proposition 4.10, we know that the sequence $(\mu_n)_n$ is bounded in $BV((0,T)\times(0,X))$ and in $L^\infty((0,T)\times(0,X))$, therefore compact in $L^p((0,T)\times(0,X))$, $1\le p<+\infty$. Therefore, a subsequence of $(\mu_n)_n$ converges in $L^p((0,T)\times(0,X))$, and the limit is nothing but μ, from the observation above. The whole sequence $(\mu_n)_n$ converges to μ in $L^p((0,T)\times(0,X))$. We have proved that $\mu_n - rx$ converges to $\mu - rx$ in $L^p((0,T)\times\mathbb{R}_+)$, $1<p<+\infty$. Finally, (5.4) follows from (5.3). ▯

6. The least square inverse problem. Let us introduce an Hilbert space H_ψ endowed with the norm $\|.\|_{H_\psi}$, relatively compact in $\mathcal{B}$.
Consider $\mathcal{H}_\psi$ a closed and convex subset of H_ψ. We assume that $\mathcal{H}_\psi$ is contained in $\left\{\psi : \|\psi\|_{\mathcal{B}}\le\bar{\psi};\ \psi\ge 0\right\}$ and that a) the functions $\psi\in\mathcal{H}_\psi$ are continuous near 0, b) there exists two positive constants $\underline{\psi}$ and $\bar{z}$ such that $\psi(z)\ge\underline{\psi}$ for all z such that $|z|\le\bar{z}$, c) there exist two constant $\underline{\zeta}>0$ and $C\ge 0$ such that for all $\psi\in\mathcal{H}_\psi$, $\psi(z)e^{3/2z}-\psi(0)e^{-3/2z} = z\omega(z)$, with $|\omega(z)|\le C|z|e^{-\zeta|z|}$, for all $z\in\mathbb{R}$. This choice of $\mathcal{H}_\psi$ will allow us to use the results stated in Lemma 3.6.
Finally, consider the set $\mathcal{H} = [\underline{\sigma},\bar{\sigma}]\times[-1/2, 1-\underline{\alpha}]\times\mathcal{H}_\psi$. Let J_R be a convex, coercive and $\mathcal{C}^1$ function defined on $[\underline{\sigma},\bar{\sigma}]\times[-1/2,1-\underline{\alpha}]\times H_\psi$. It is well known that J_R is also weakly lower semicontinuous. The functional J_R may depend on suitable prior parameters σ_0, α_0 and ψ_0. It is the analog of the function ρJ_2 discussed in § 1.

6.1. Toward the calibration problem.

6.1.1. Orientation. For calibrating the Lévy process, one observes the spot price S and the prices $(\bar{p}_i)_{i\in I}$ of a family of American put options with maturities/strikes given by (T_i, x_i), and we call $\bar{u}_i = \bar{p}_i - x_i + S$, $i\in I$. The parameters of the Lévy process, i.e. the volatility σ, the exponent α and the function ψ will be found as solutions of a least square problem, where the functional to be minimized is the sum of a suitable Tychonoff regularization functional and of

$$J(u) = \sum_{i\in I}\omega_i(u(T_i,x_i)-\bar{u}_i)^2,$$

where ω_i are positive weights, and $u = u(\sigma,\alpha,\psi)$ is a solution of (VIP).
We aim at finding some necessary optimality conditions satisfied by the solutions of the least square problem. The main difficulty comes from the fact that the derivability of the functional $J(u)$ with respect to the parameter (σ,α,ψ) is not guaranteed. To obtain some necessary optimality conditions, we shall consider first a least square problem where u is the solution of the penalized problem (4.6) rather than (VIP), obtain the necessary optimality conditions for this new problem, then have the penalty parameter ϵ tend to 0 and pass to the limit in the optimality conditions. Such a program has already been applied in [2] for calibrating the local volatility with American options, see also [4, 5] for a related numerical method and results. The idea originally comes from Hintermüller [22] and Ito and Kunisch[23], who applied a similar program for elliptic variational inequalities. Let us also mention Mignot and Puel [32] who applied a nice method for finding the optimality conditions of a special control problem with a parabolic variational inequality.
In order to simplify the notations, we are going to consider first a toy problem where only one price is observed. Of course, observing one price only is not enough. However, finding the optimality conditions for this simplified calibration problem presents the same difficulties as for the original one.

6.1.2. The least square problem and its penalized version. A first step towards the calibration problem is to consider the functional J

$$J:\mathcal{C}^0([0,T]\times\mathbb{R}_+)\to\mathbb{R},\qquad J(u) = (u(T,x_{ob})-\bar{u})^2, \tag{6.1}$$

where x_{ob} and $\bar{u}$ are positive numbers. We fix $\bar{X}$ (independent of $(\sigma, \alpha, \psi) \in \mathcal{H}$) as in Proposition 4.12 and assume that $x_{ob} < \bar{X}$. Consider the least square problem:

$$\text{Minimize } J(u)+J_R(\sigma,\alpha,\psi) \;\Big|\; (\sigma,\alpha,\psi) \in \mathcal{H}, u = u(\sigma,\alpha,\psi) \text{ satisfies (VIP)} . \quad (6.2)$$

Fixing $X \geq \bar{X}$, we know that $u|_{[0,T]\times[0,X]} = u_X$ where u_X is the solution of (VIP$_X$). Therefore, (6.2) is equivalent to the least square problem:

$$\text{Minimize } J(u)+J_R(\sigma,\alpha,\psi) \;\Big|\; (\sigma,\alpha,\psi) \in \mathcal{H}, u \text{ satisfies (VIP}_X) . \quad (6.3)$$

We will also consider the least square problem related to the penalized problem

$$\text{Minimize } J(u_\epsilon)+J_R(\sigma,\alpha,\psi) \;\Big|\; (\sigma,\alpha,\psi) \in \mathcal{H}, u_\epsilon \text{ satisfies (4.6)} . \quad (6.4)$$

LEMMA 6.1. *Let $(\epsilon_n)_n$ be a sequence of penalty parameters such that $\epsilon_n \to 0$ as $n \to \infty$, and let $(\sigma^*_{\epsilon_n}, \alpha^*_{\epsilon_n}, \psi^*_{\epsilon_n}), u^*_{\epsilon_n}$ be a solution of problem (6.4). Consider a subsequence such that $(\sigma^*_{\epsilon_n}, \alpha^*_{\epsilon_n}, \psi^*_{\epsilon_n})$ converges to $(\sigma^*, \alpha^*, \psi^*)$ in $\mathcal{F}$, $\psi^*_{\epsilon_n}$ weakly converges to ψ^* in H_ψ and $u^*_{\epsilon_n} \to u^*$ weakly in $L^2(0,T;V_X)$, where V_X is defined in (4.1). Then $(\sigma^*, \alpha^*, \psi^*), u^*$ is a solution of (6.3). We have that*
- *$u^*_{\epsilon_n}$ converges to u^* uniformly in $[0,T] \times [0,X]$, and in $L^2(0,T;V_X)$.*
- *$1_{\{x>S\}} rx\mathcal{V}_{\epsilon_n}(u^*_{\epsilon_n})$ converges to μ^* strongly in $L^2((0,T)\times(0,X))$,*
- *For all smooth functions χ with compact support contained in $[0,X)$, $\chi\mathcal{E}_X(u^*_{\epsilon_n})$ converges to $\chi\mathcal{E}_X(u^*)$ strongly in $L^2(0,T;V^2)$ and in $L^\infty(0,T;V)$.*

Proof. For brevity, the proof is outlined only. We skip the proof that u^* satisfies (VIP$_X$) with $(\sigma, \alpha, \psi) = (\sigma^*, \alpha^*, \psi^*)$ and the proofs of the first two points above, since they are in the same spirit as the proofs of Theorem 4.4 and Proposition 4.8. The third point above is proved by writing the boundary value problems satisfied by $y_n = \chi\mathcal{E}_X(u^*_{\epsilon_n})$ and $y = \chi\mathcal{E}_X(u^*)$, with the PID equations

$$\frac{\partial y_n}{\partial t} + A_n y_n = f_n, \qquad \frac{\partial y}{\partial t} + Ay = f,$$

where A (resp. A_n) is given by (3.9) and (2.12) with $(\sigma, \alpha, \psi) = (\sigma^*, \alpha^*, \psi^*)$ (resp. $(\sigma, \alpha, \psi) = (\sigma^*_{\epsilon_n}, \alpha^*_{\epsilon_n}, \psi^*_{\epsilon_n})$) and where the right hand side f (resp. f_n) can be written in terms of χ,u^* and μ^* (resp. χ and $u^*_{\epsilon_n}$). By using the first two points above and the same arguments as in the proofs of Propositions 5.1 and 5.2, it can be proved that f_n converges to f in $L^2((0,T)\times\mathbb{R}_+)$, and that y_n converges to y in $L^2(0,T;V^2)$ and in $L^\infty(0,T;V)$.
As a consequence of the first point above, $J(u^*_{\epsilon_n}) \to J(u^*)$. Moreover, from the assumptions on J_R, $J_R(\sigma^*, \alpha^*, \psi^*) \leq \liminf_{n\to\infty} J_R(\sigma^*_{\epsilon_n}, \alpha^*_{\epsilon_n}, \psi^*_{\epsilon_n})$.
Since $(\sigma^*_{\epsilon_n}, \alpha^*_{\epsilon_n}, \psi^*_{\epsilon_n})$ is a solution of (6.4),

$$J(u^*_{\epsilon_n}) + J_R(\sigma^*_{\epsilon_n}, \alpha^*_{\epsilon_n}, \psi^*_{\epsilon_n}) \leq J(u_{\epsilon_n}(\sigma,\alpha,\psi)) + J_R(\sigma,\alpha,\psi), \quad \forall(\sigma,\alpha,\psi) \in \mathcal{H},$$

where $u_{\epsilon_n}(\sigma, \alpha, \psi)$ is the solution of (4.6) with $\epsilon = \epsilon_n$. This implies that

$$J(u^*) + J_R(\sigma^*, \alpha^*, \psi^*) \leq J(u(\sigma,\alpha,\psi)) + J_R(\sigma,\alpha,\psi), \quad \forall(\sigma,\alpha,\psi) \in \mathcal{H},$$

where $u(\sigma, \alpha, \psi)$ satisfies (VIP$_X$) and $(\sigma^*, \alpha^*, \psi^*), u^*$ is a solution of (6.3). □

Remark 10. *Let $(\sigma^*_{\epsilon_n}, \alpha^*_{\epsilon_n}, \psi^*_{\epsilon_n}), u^*_{\epsilon_n}$ be a subsequence converging to $(\sigma^*, \alpha^*, \psi^*)$, u^* as in Lemma 6.1. It is clear from the continuity of u^* and from the uniform convergence of $u^*_{\epsilon_n}$ that if $u^*(T, x_{ob}) > u_\circ(x_{ob})$, then there exists a constant $a > 0$ and an integer N such that for $n > N$, $u^*_{\epsilon_n}(t,x) > u_\circ(x) + \epsilon_n$ for all (t,x) with $|x - x_{ob}| < a$ and $t > T - a$.*

6.1.3. First order necessary optimality conditions for (6.4). We take $(\sigma^*_{\epsilon_n}, \alpha^*_{\epsilon_n}, \psi^*_{\epsilon_n}), u^*_{\epsilon_n}$ and $(\sigma^*, \alpha^*, \psi^*)$, u^* as in Lemma 6.1. We assume that $u^*(T, x_{ob}) > u_\circ(x_{ob})$ and we take N and a as in Remark 10. For $n > N$, we wish to find necessary optimality conditions for the solution $(\sigma^*_{\epsilon_n}, \alpha^*_{\epsilon_n}, \psi^*_{\epsilon_n}), u^*_{\epsilon_n}$ of (6.4). In order to simplify the notations, we drop the index n: below, ϵ means ϵ_n.
We shall need to solve an adjoint problem. Since the cost functional involves pointwise values of u, the adjoint problem will have a singular data. In that context, the notion of very weak solution of boundary value problems will be relevant: we introduce the space $Z_\epsilon = \left\{ v \in \tilde{Z}_\epsilon; v(t=0) = 0 \right\}$, where

$$\tilde{Z}_\epsilon = \left\{ v \in L^2(0,T;V_X); \frac{\partial v}{\partial t} + A_{\epsilon,X} v - rx1_{\{x>S\}} \mathcal{V}'(u^*_\epsilon) v \in L^2((0,T)\times(0,X)) \right\}$$

and $A_{\epsilon,X}$ is the operator defined by (4.2), with $(\sigma, \alpha, \psi) = (\sigma^*_\epsilon, \alpha^*_\epsilon, \psi^*_\epsilon)$. The space Z_ϵ endowed with the graph norm is a Banach space.

LEMMA 6.2. *Assume that $u^*(T, x_{ob}) > u_\circ(x_{ob})$ and take N and a as in Remark 10. There exists a unique $p^*_\epsilon \in L^2((0,T)\times(0,X))$ such that, for all $v \in Z_\epsilon$,*

$$\int_0^T \int_0^X \left(\frac{\partial v}{\partial t} + A_{\epsilon,X} v - rx1_{\{x>S\}} \mathcal{V}'_\epsilon(u^*_\epsilon) v \right) p^*_\epsilon = 2(u^*_\epsilon(T, x_{ob}) - \bar{u}) v(T, x_{ob}), \quad (6.5)$$

and $\|p^*_\epsilon\|_{L^2((0,T)\times(0,X))}$ is bounded by a constant independent of ϵ in the subsequence. For a fixed smooth function ϕ taking the value 1 for $|x - x_{ob}| \geq a/2$, $T - t \geq a/2$ and vanishing in a neighborhood of (T, x_{ob}), we have that $\phi p^*_\epsilon \in L^2(0,T;V_X) \cap \mathcal{C}^0([0,T]; L^2((0,X)))$, with norms bounded independently of ϵ.

Proof. See § 8. ☐

Remark 11. *Problem (6.5) is a very weak formulation of:*

$$\begin{aligned}
&\frac{\partial p^*_\epsilon}{\partial t} - A^T_{\epsilon,X} p^*_\epsilon + rx1_{\{x>S\}} \mathcal{V}'_\epsilon(u^*_\epsilon) p^*_\epsilon = 0, \qquad (t,x) \in [0,T) \times (0,X),\\
&p^*_\epsilon(t, X) = 0, t \in (0,T),\\
&p^*_\epsilon(T) = -2(u^*_\epsilon(T, x_{ob}) - \bar{u}) \delta_{x_{ob}},
\end{aligned} \quad (6.6)$$

where $A^T_{\epsilon,X} v(x) = -\frac{(\sigma^*_\epsilon)^2}{2} \frac{\partial^2}{\partial x^2}(x^2 v) + B^T_{\epsilon,X} v(x) - \frac{\partial}{\partial x}(rxv)$ with

$$B^T_{\epsilon,X} v(x) = \int_{\mathbb{R}} k^*_\epsilon(z) \left(x(e^z - 1) \frac{\partial v}{\partial x}(x) - 1_{z<\log\frac{X}{x}} e^{2z} v(xe^z) + (2e^z - 1) v(x) \right) dz,$$

and $k^*_\epsilon(z) = |z|^{-(2\alpha^*_\epsilon + 1)} \psi^*_\epsilon(z)$.

Remark 12. *Similarly, $\left\| \frac{\partial(\phi p^*_\epsilon)}{\partial t} - A^T_{\epsilon,X}(\phi p^*_\epsilon) + rx1_{\{x>S\}} \mathcal{V}'_\epsilon(u^*_\epsilon)(\phi p^*_\epsilon) \right\|_{L^2((0,T)\times(0,X))}$ is bounded independently of ϵ.*

From lemma 6.2, we see that $\int_0^T \left\langle x^2 \frac{\partial^2 u^*_\epsilon}{\partial x^2}, \phi p^*_\epsilon \right\rangle$ is well defined, where $\langle , \rangle$ is the duality pairing between $(V_X)'$ and V_X. On the other hand, $\int_0^T \int_0^X \left((1-\phi) x^2 \frac{\partial^2 u^*_\epsilon}{\partial x^2} \right) p^*_\epsilon$ is well defined since both $\left((1-\phi) x^2 \frac{\partial^2 u^*_\epsilon}{\partial x^2} \right)$ and p^*_ϵ are square integrable. Moreover, the sum $\int_0^T \left\langle x^2 \frac{\partial^2 u^*_\epsilon}{\partial x^2}, \phi p^*_\epsilon \right\rangle + \int_0^T \int_0^X \left((1-\phi) x^2 \frac{\partial^2 u^*_\epsilon}{\partial x^2} \right) p^*_\epsilon$ does not depend on the choice of ϕ. Therefore, we call $\mathcal{G}^{(\sigma)}(u^*_\epsilon, p^*_\epsilon)$ the quantity

$$\mathcal{G}^{(\sigma)}(u^*_\epsilon, p^*_\epsilon) = \int_0^T \left\langle x^2 \frac{\partial^2 u^*_\epsilon}{\partial x^2}, \phi p^*_\epsilon \right\rangle + \int_0^T \int_0^X \left((1-\phi) x^2 \frac{\partial^2 u^*_\epsilon}{\partial x^2} \right) p^*_\epsilon. \quad (6.7)$$

Let us introduce the operator $B^{(\alpha)}_{\epsilon,X}$:

$$\begin{aligned}
&B^{(\alpha)}_{\epsilon,X} v(x) =\\
&-\int_{\mathbb{R}} k^*_\epsilon(z) \log(|z|) \left(x(e^z - 1) \frac{\partial v}{\partial x}(x) + e^z (1_{\{z>-\log(\frac{X}{x})\}} v(xe^{-z}) - v(x)) \right) dz,
\end{aligned} \quad (6.8)$$

where $k^*_\epsilon(z) = |z|^{-2\alpha^*_\epsilon-1}\psi^*_\epsilon(z)$. From Lemma 6.2, the quantity $\int_0^T \left\langle B^{(\alpha)}_{\epsilon,X}u^*_\epsilon, \phi p^*_\epsilon\right\rangle$ is well defined, where $\langle , \rangle$ is the duality pairing between $(V_X)'$ and V_X. On the other hand, the quantity $\int_0^T\int_0^X \left((1-\phi)B^{(\alpha)}_{\epsilon,X}u^*_\epsilon\right)p^*_\epsilon$ is well defined since both p^*_ϵ and $\left((1-\phi)B^{(\alpha)}_{\epsilon,X}u^*_\epsilon\right)$ are square integrable. Moreover, the sum $\int_0^T \left\langle B^{(\alpha)}_{\epsilon,X}u^*_\epsilon, \phi p^*_\epsilon\right\rangle + \int_0^T\int_0^X \left((1-\phi)B^{(\alpha)}_{\epsilon,X}u^*_\epsilon\right)p^*_\epsilon$ does not depend on ϕ. Therefore, we denote $\mathcal{G}^{(\alpha)}_\epsilon(u^*_\epsilon, p^*_\epsilon)$ the quantity

$$\mathcal{G}^{(\alpha)}_\epsilon(u^*_\epsilon, p^*_\epsilon) = \int_0^T \left\langle B^{(\alpha)}_{\epsilon,X}u^*_\epsilon, \phi p^*_\epsilon\right\rangle + \int_0^T\int_0^X \left((1-\phi)B^{(\alpha)}_{\epsilon,X}u^*_\epsilon\right)p^*_\epsilon. \tag{6.9}$$

Similarly, for $\kappa \in H_\psi$, we introduce the operator $B^{(\psi,\kappa)}_{\epsilon,X}$:

$$B^{(\psi,\kappa)}_{\epsilon,X}v(x) = \int_{\mathbb{R}} \frac{\kappa(z)}{|z|^{1+2\alpha^*_\epsilon}}\left(x(e^z-1)\frac{\partial v}{\partial x}(x) + e^z(1_{\{z>-\log(\frac{X}{x})\}}v(xe^{-z}) - v(x))\right)dz,$$

and the quantity

$$\left\langle \mathcal{G}^{(\psi)}_\epsilon(u^*_\epsilon, p^*_\epsilon), \kappa\right\rangle = \int_0^T \left\langle B^{(\psi,\kappa)}_{\epsilon,X}u^*_\epsilon, \phi p^*_\epsilon\right\rangle + \int_0^T\int_0^X \left((1-\phi)B^{(\psi,\kappa)}_{\epsilon,X}u^*_\epsilon\right)p^*_\epsilon, \tag{6.10}$$

which does not depend on ϕ. We are now ready to give necessary optimality for the least square problem (6.4):

Proposition 6.3. *The optimality conditions for problem (6.4) are: for all $(\sigma, \alpha, \psi) \in \mathcal{H}$,*

$$(\sigma - \sigma^*_\epsilon)\left(D_\sigma J_R(\sigma^*_\epsilon, \alpha^*_\epsilon, \psi^*_\epsilon) + \sigma^*_\epsilon \mathcal{G}^{(\sigma)}(u^*_\epsilon, p^*_\epsilon)\right) \geq 0, \tag{6.11}$$

$$(\alpha - \alpha^*_\epsilon)\left(D_\alpha J_R(\sigma^*_\epsilon, \alpha^*_\epsilon, \psi^*_\epsilon) + 2\mathcal{G}^{(\alpha)}_\epsilon(u^*_\epsilon, p^*_\epsilon)\right) \geq 0, \tag{6.12}$$

$$\langle D_\psi J_R(\sigma^*_\epsilon, \alpha^*_\epsilon, \psi^*_\epsilon), \psi - \psi^*_\epsilon\rangle + \left\langle \mathcal{G}^{(\psi)}_\epsilon(u^*_\epsilon, p^*_\epsilon), \psi - \psi^*_\epsilon\right\rangle \geq 0. \tag{6.13}$$

Proof. The proof is quite standard. It is omitted for brevity. ∎

6.1.4. First order necessary optimality conditions for (6.3). In order to obtain optimality conditions for (6.3), we wish to pass to the limit in the optimality conditions for (6.4). Let ϵ_n be sequence of penalty parameters converging to zero, and let $(\sigma^*_{\epsilon_n}, \alpha^*_{\epsilon_n}, \psi^*_{\epsilon_n}, u^*_{\epsilon_n})$ be a sequence of solutions to (6.4) converging to $(\sigma^*, \alpha^*, \psi^*, u^*)$ as in Lemma 6.1. Assume that there exists a positive number a such that $u^*_{\epsilon_n}(t,x) > u_\circ(x) + \epsilon_n$ for all (t,x) with $|x - x_{ob}| \leq a$ and $T - t \leq a$. Let $p^*_{\epsilon_n}$ be the adjoint state defined by Lemma 6.2. There exists a subsequence denoted n_k such that $p^*_{\epsilon_{n_k}}$ weakly converges to p^* in $L^2((0,T)\times(0,X))$ and $\phi p^*_{\epsilon_{n_k}}$ weakly converges to ϕp^* in $L^2(0,T;V_X)$, where ϕ is given in Lemma 6.2.
We call $\tilde{Z}$ and Z the spaces

$$\begin{aligned} \tilde{Z} &= \left\{v \in L^2(0,T;V_X); \frac{\partial v}{\partial t} + A_X v \in L^2((0,T)\times(0,X))\right\}, \\ Z &= \left\{v \in \tilde{Z}; v(t=0) = 0\right\}, \end{aligned} \tag{6.14}$$

where A_X is the operator given by (4.2), (3.9) and (2.12), with the parameter $(\sigma^*, \alpha^*, \psi^*)$. These spaces, endowed with the graph norm, are Banach spaces.

Proposition 6.4. *There exists a Radon measure ξ^* such that for all $v \in Z$,*

$$\int_0^T\int_0^X \left(\frac{\partial v}{\partial t} + A_X v\right)p^* + < \xi^*, v > = 2(u^*(T,x_{ob}) - \bar{u})v((T,x_{ob})). \tag{6.15}$$

The function p^* satisfies

$$\frac{\partial p^*}{\partial t} - A_X^T p^* - \xi^* = 0 \tag{6.16}$$

in the sense of distributions. Furthermore, with u^*, μ^* defined as in Lemma 6.1,

$$\mu^* |p^*| = 0, \tag{6.17}$$

$$|u^*| \xi^* = 0. \tag{6.18}$$

Proof. For simplicity, we drop the index n in ϵ_n. In what follows, ϵ means ϵ_n. For a positive parameter δ, we introduce the nondecreasing function $\rho_\delta : \mathbb{R} \to \mathbb{R}$:

$$\rho_\delta(p) = -1 \text{ for } p \le -\delta, \quad \rho_\delta(p) = p/\delta \text{ for } -\delta \le p \le \delta, \quad \rho_\delta(p) = 1 \text{ for } p \ge \delta.$$

and the nonnegative function $R_\delta(p) = \int_0^p \rho_\delta(q)dq$.
In what follows, δ will be the generic term of a decreasing sequence of positive parameters which converges to 0.
For ϕ introduced in Lemma 6.2, we use Remark 12: there exists a function $g_\epsilon \in L^2((0,T)\times(0,X))$ with a norm bounded independently of ϵ such that ϕp_ϵ^* is the weak solution to

$$\frac{\partial(\phi p_\epsilon^*)}{\partial t} - A_{\epsilon,X}^T(\phi p_\epsilon^*) + rx 1_{\{x>S\}} \mathcal{V}_\epsilon'(u_\epsilon^*)(\phi p_\epsilon^*) = g_\epsilon$$

with the Cauchy condition $(\phi p_\epsilon^*)(T,.) = 0$ and the boundary condition $(\phi p_\epsilon^*)(.,X) = 0$. Therefore, $\|\phi p_\epsilon^*\|_{L^2(0,T;V_X)}$ is bounded uniformly in ϵ. Moreover, from the properties of ϕ, see Lemma 6.2), we have $\mathcal{V}_\epsilon'(u_\epsilon^*)(\phi p_\epsilon^*) = \mathcal{V}_\epsilon'(u_\epsilon^*)p_\epsilon^*$.
Multiplying the last equation by $\rho_\delta(\phi p_\epsilon^*)$, we obtain that there exists a constant C independent of δ and ϵ such that

$$\begin{aligned} &\int_0^X R_\delta(\phi p_\epsilon^*)(0,x)dx + \int_0^T \int_0^X \frac{(\sigma_\epsilon^*)^2 x^2}{2} \rho_\delta'(\phi p_\epsilon^*)(\frac{\partial(\phi p_\epsilon^*)}{\partial x})^2 \\ &-r \int_0^T \int_S^X x \mathcal{V}_\epsilon'(u_\epsilon^*) p_\epsilon^* \rho_\delta(p_\epsilon^*) + \int_0^T \left\langle \left(B_{\epsilon,X}^T(\phi p_\epsilon^*)\right), \rho_\delta(\phi p_\epsilon^*)\right\rangle \le C. \end{aligned} \tag{6.19}$$

Let us focus on the last term in the sum above: we can write it as

$$\begin{aligned} \int_0^T \int_{\mathbb{R}_+} \left(B_\epsilon^T(\mathcal{E}_X(\phi p_\epsilon^*))\right) \rho_\delta(\mathcal{E}_x(\phi p_\epsilon^*)) = &\frac{1}{2}\int_0^T \left\langle (B_\epsilon + B_\epsilon^T)(\mathcal{E}_X(\phi p_\epsilon^*)), \rho_\delta(\mathcal{E}_x(\phi p_\epsilon^*))\right\rangle \\ &- \frac{1}{2}\int_0^T \left\langle (B_\epsilon - B_\epsilon^T)(\mathcal{E}_X(\phi p_\epsilon^*)), \rho_\delta(\mathcal{E}_x(\phi p_\epsilon^*))\right\rangle. \end{aligned} \tag{6.20}$$

From Lemma 3.6 and the choice of $\mathcal{H}_\psi$, there exists a constant C independent of $(\sigma,\alpha,\psi) \in \mathcal{H}$ and of δ such that

$$\begin{aligned} &\left| \int_0^T \left\langle (B_\epsilon - B_\epsilon^T)(\mathcal{E}_X(\phi p_\epsilon^*)), \mathcal{E}_X(\rho_\delta(\phi p_\epsilon^*))\right\rangle \right| \\ &\lesssim \|\phi p_\epsilon^*\|_{L^2((0,T);V_X)} \|\rho_\delta(\phi p_\epsilon^*)\|_{L^2((0,T)\times(0,X))} \le C. \end{aligned} \tag{6.21}$$

On the other hand, from Lemma 3.5,

$$\begin{aligned} &\left| \begin{array}{l} \int_0^T \left\langle (B_\epsilon + B_\epsilon^T)(\mathcal{E}_X(\phi p_\epsilon^*)), \mathcal{E}_X(\rho_\delta(\phi p_\epsilon^*))\right\rangle \\ - \int_0^T \int_0^X \int_{\mathbb{R}} k_\epsilon(z) e^z \left((\phi p_\epsilon^*)(x) - (\phi p_\epsilon^*)(xe^{-z})\right) \left(\rho_\delta(\phi p_\epsilon^*)(x) - \rho_\delta(\phi p_\epsilon^*)(xe^{-z})\right) \end{array} \right| \\ &\lesssim \|\phi p_\epsilon^*\|_{L^2((0,T)\times(0,X))} \|\rho_\delta(\phi p_\epsilon^*)\|_{L^2((0,T)\times(0,X))} \le C. \end{aligned} \tag{6.22}$$

From (6.19), (6.20), (6.21) and (6.22), we see that

$$\int_0^X R_\delta(\phi p_\epsilon^*)(0,x)dx$$
$$+\int_0^T\int_0^X\int_{\mathbb{R}} k_\epsilon(z)e^z\left((\phi p_\epsilon^*)(x)-(\phi p_\epsilon^*)(xe^{-z})\right)\left(\rho_\delta(\phi p_\epsilon^*)(x)-\rho_\delta(\phi p_\epsilon^*)(xe^{-z})\right)$$
$$+\int_0^T\int_0^X \frac{(\sigma_\epsilon^*)^2x^2}{2}\rho_\delta'(\phi p_\epsilon^*)(\frac{\partial(\phi p_\epsilon^*)}{\partial x})^2 - r\int_0^T\int_S^X x\mathcal{V}_\epsilon'(u_\epsilon^*)p_\epsilon^*\rho_\delta(p_\epsilon^*)\leq C.$$

Since $p\mapsto p\rho_\delta(p)$ is a nonnegative function and since ρ_δ is nondecreasing, all the terms in the sum above are nonnegative. Therefore, for a constant C independent of the parameters, $-r\int_0^T\int_S^X x\mathcal{V}_\epsilon'(u_\epsilon^*)p_\epsilon^*\rho_\delta(p_\epsilon^*)\leq C$.
On the other hand we know that $-x\mathcal{V}_\epsilon'(u_\epsilon^*)p_\epsilon^*\rho_\delta(p_\epsilon^*)$ defines an increasing (as δ decreases) sequence of nonnegative functions, which converges almost everywhere to $x|\mathcal{V}_\epsilon'(u_\epsilon^*)p_\epsilon^*|$ as δ tends to 0. Thus, Beppo Levi's theorem tells us that $-r\int_0^T\int_S^X x\mathcal{V}_\epsilon'(u_\epsilon^*)p_\epsilon^*\rho_\delta(p_\epsilon^*)$ tends to $r\int_0^T\int_S^X x|\mathcal{V}_\epsilon'(u_\epsilon^*)p_\epsilon^*|$ as $\delta\to 0$. Therefore, for a positive constant C,

$$r\int_0^T\int_S^X x|\mathcal{V}_\epsilon'(u_\epsilon^*)p_\epsilon^*|\leq C. \tag{6.23}$$

It is thus possible to extract a subsequence ϵ_{n_k}, such that $p_{\epsilon_{n_k}}^*\to p^*$ weakly in $L^2((0,T)\times(0,X))$, $\phi p_{\epsilon_{n_k}}^*\to\phi p^*$ weakly in $L^2(0,T;V_X)$, and $-rx1_{\{x>S\}}\mathcal{V}_{\epsilon_{n_k}}'(u_{\epsilon_{n_k}}^*)p_{\epsilon_{n_k}}^*$ converges to ξ^* weakly* in $(L^\infty((0,T)\times(0,X)))^*$. In order to simplify the notations, we omit the indexes n_k: now, ϵ means ϵ_{n_k}.
From this, (6.15) is obtained as well by passing to the limit in (6.5), and (6.16) is satisfied in the sense of distributions.
For proving (6.17), we use the convexity of $\mathcal{V}_\epsilon$, (still dropping the index n_k in ϵ_{n_k}): since $\mathcal{V}_\epsilon(\epsilon)=0$, we have that for all $u\in[0,\epsilon]$, $\mathcal{V}_\epsilon(u)\leq-\mathcal{V}_\epsilon'(u)(\epsilon-u)\leq-\epsilon\mathcal{V}_\epsilon'(u)$. This implies that $\mathcal{V}_\epsilon(u_\epsilon^*)\leq-\epsilon\mathcal{V}_\epsilon'(u_\epsilon^*)$ because we also know that $\mathcal{V}_\epsilon(u_\epsilon^*)=0$ if $u_\epsilon^*\geq\epsilon$. Thus, calling $\mu_\epsilon^*=rx1_{\{x>S\}}\mathcal{V}_\epsilon(u_\epsilon^*)$, (6.23) implies that

$$0\leq\int_0^T\int_0^X\mu_\epsilon^*|p_\epsilon^*|\leq-\epsilon\, r\int_0^T\int_S^X x\mathcal{V}_\epsilon'(u_\epsilon^*)|p_\epsilon^*|\to 0. \tag{6.24}$$

But we also know that $p_\epsilon^*\to p^*$ weakly in $L^2((0,T)\times(0,X))$ and that $\mu_\epsilon^*\to\mu^*$ strongly in $L^2((0,T)\times(0,X))$ from Lemma 6.1. Hence, $\int_0^T\int_0^X\mu_\epsilon^*|p_\epsilon^*|\to\int_0^T\int_0^X\mu^*|p^*|$, and (6.17) is proved.
Let us call $\xi_\epsilon^*=-rx1_{\{x>S\}}\ \mathcal{V}_\epsilon'(u_\epsilon^*)p_\epsilon^*=-rx1_{\{x>S\}}\ \mathcal{V}_\epsilon'(u_\epsilon^*)\phi p_\epsilon^*$; for χ a continuous function in $[0,T]\times[0,X]$, we have

$$\int_0^T\int_0^X|\xi_\epsilon^*||\chi u_\epsilon^*|\leq r\left(\int_0^T\int_S^X|x\mathcal{V}_\epsilon'(u_\epsilon^*)||\phi p_\epsilon^*|^2\right)^{\frac{1}{2}}\left(\int_0^T\int_S^X|x\mathcal{V}_\epsilon'(u_\epsilon^*)||\chi u_\epsilon^*|^2\right)^{\frac{1}{2}}$$

from the Cauchy-Schwarz inequality. But it can be checked that $|\mathcal{V}_\epsilon'(u_\epsilon^*)||u_\epsilon^*\chi|^2\leq C\epsilon$, which yields $\int_0^T\int_S^X|x\mathcal{V}_\epsilon'(u_\epsilon^*)||u_\epsilon^*\chi|^2\leq C\epsilon$. On the other hand, it is easy to check that $\int_0^T\int_S^X|x\mathcal{V}_\epsilon'(u_\epsilon^*)||\phi p_\epsilon^*|^2\leq C$. Therefore $\int_0^T\int_{\mathbb{R}_+}\xi_\epsilon^*|u_\epsilon^*|\chi\to 0$ as $\epsilon\to 0$. We know that $\xi_\epsilon^*\to\xi^*$ weakly in $(L^\infty)^*$ and that $|u_\epsilon^*|\chi\to|u^*|\chi$ in $C^0([0,T]\times[0,X])$ from Lemma 6.1. We can pass to the limit as $\epsilon\to 0$ and (6.18) is proved. ◻

Proceeding as in (6.7), (6.9), (6.10), we introduce the quantities

$$\mathcal{G}^{(\sigma)}(u^*,p^*) = \int_0^T \left\langle x^2\frac{\partial^2 u^*}{\partial x^2}, \phi p^*\right\rangle + \int_0^T\int_0^X \left((1-\phi)x^2\frac{\partial^2 u^*}{\partial x^2}\right)p^*, \quad (6.25)$$

$$\mathcal{G}^{(\alpha)}(u^*,p^*) = \int_0^T \left\langle B_X^{(\alpha)}u^*, \phi p^*\right\rangle + \int_0^T\int_0^X \left((1-\phi)B_X^{(\alpha)}u^*\right)p^*, \quad (6.26)$$

$$\left\langle \mathcal{G}^{(\psi)}(u^*,p^*),\kappa\right\rangle = \int_0^T \left\langle B_X^{(\psi,\kappa)}u^*, \phi p^*\right\rangle + \int_0^T\int_0^X \left((1-\phi)B_X^{(\psi,\kappa)}u^*\right)p^*, \quad (6.27)$$

where ϕ is chosen as in Lemma 6.2, and where

$$B_X^{(\alpha)}v(x) = -\int_{\mathbb{R}} k^*(z)\log(|z|)\left(x(e^z-1)\frac{\partial v}{\partial x}(x) + e^z(1_{\{z>-\log(\frac{X}{x})\}}v(xe^{-z})-v(x))\right),$$

$$B_X^{(\psi,\kappa)}v(x) = \int_{\mathbb{R}} \frac{\kappa(z)}{|z|^{1+2\alpha^*}}\left(x(e^z-1)\frac{\partial v}{\partial x}(x) + e^z(1_{\{z>-\log(\frac{X}{x})\}}v(xe^{-z})-v(x))\right)dz.$$

One can check exactly as above that $\mathcal{G}^{(\sigma)}(u^*,p^*)$, $\mathcal{G}^{(\alpha)}(u^*,p^*)$ and $\left\langle \mathcal{G}^{(\psi)}(u^*,p^*),\kappa\right\rangle$ are well defined and do not depend of the particular choice of ϕ. We are now ready to give necessary optimality for the least square problem (6.4):

Proposition 6.5. *Let $(\sigma^*,\alpha^*,\psi^*,u^*)$ be a solution to problem (6.3) obtained in Lemma 6.1. Assume that $u^*(T,x_{ob}) > u_\circ(x_{ob})$ and take a as in Remark 10. There exist $p^*\in L^2((0,T)\times(0,X))$ and a Radon measure ξ^* satisfying (6.15), (6.17), (6.18), and such that, for all $(\sigma,\alpha,\psi)\in\mathcal{H}$,*

$$(\sigma-\sigma^*)\left(D_\sigma J_R(\sigma^*,\alpha^*,\psi^*) + \sigma^*\mathcal{G}^{(\sigma)}(u^*,p^*)\right) \geq 0, \quad (6.28)$$

$$(\alpha-\alpha^*)\left(D_\alpha J_R(\sigma^*_\epsilon,\alpha^*_\epsilon,\psi^*_\epsilon) + 2\mathcal{G}^{(\alpha)}(u^*,p^*)\right) \geq 0, \quad (6.29)$$

$$\langle D_\psi J_R(\sigma^*_\epsilon,\alpha^*_\epsilon,\psi^*_\epsilon),\psi-\psi^*\rangle + \left\langle \mathcal{G}^{(\psi)}(u^*,p^*),\psi-\psi^*\right\rangle \geq 0. \quad (6.30)$$

Proof. We consider a sequence of parameters ϵ_n such that
1) $(\sigma^*_{\epsilon_n},\alpha^*_{\epsilon_n},\psi^*_{\epsilon_n},u^*_{\epsilon_n})$ is a sequence of solutions to (6.4) converging to $(\sigma^*,\alpha^*,\psi^*,u^*)$ as in Lemma 6.1,
2) $u^*_{\epsilon_n}(t,x) > u_\circ(x)+\epsilon_n$ for all (t,x) with $|x-x_{ob}|\leq a$ and $T-t\leq a$,
3) for the adjoint states p^*_ϵ defined by Lemma 6.2, $p^*_{\epsilon_n}$ weakly converges to p^* in $L^2((0,T)\times(0,X))$ and $\phi p^*_{\epsilon_n}$ weakly converges to ϕp^* in $L^2(0,T;V_X)$, where ϕ is given in Lemma 6.2.
We drop the index n in ϵ_n. We have to prove that $\lim_{\epsilon\to 0}\mathcal{G}^{(\sigma)}(u^*_\epsilon,p^*_\epsilon) = \mathcal{G}^{(\sigma)}(u^*,p^*)$. Since $u^*_\epsilon\to u^*$ strongly in $L^2(0,T;V_X)$ (see Lemma 6.1) and $\phi p^*_\epsilon\to\phi p^*$ weakly in $L^2(0,T;V_X)$, we deduce that

$$\lim_{\epsilon\to 0}\int_0^T \left\langle \frac{\partial^2 u^*_\epsilon}{\partial^2 x},\phi p^*_\epsilon\right\rangle = \int_0^T \left\langle \frac{\partial^2 u^*}{\partial^2 x},\phi p^*\right\rangle.$$

On the other hand, $(1-\phi)\frac{\partial^2 u^*_\epsilon}{\partial^2 x}$ strongly converges to $(1-\phi)\frac{\partial^2 u^*}{\partial^2 x}$ in $L^2((0,T)\times(0,X))$ from Lemma 6.1, and p^*_ϵ weakly converges to p^* in $L^2((0,T)\times(0,X))$. Thus

$$\lim_{\epsilon\to 0}\int_0^T\int_0^X \left((1-\phi)\frac{\partial^2 u^*_\epsilon}{\partial^2 x}\right)p^*_\epsilon = \int_0^T\int_0^X \left((1-\phi)\frac{\partial^2 u^*}{\partial^2 x}\right)p^*.$$

From the two points above, we see that $\lim_{\epsilon\to 0}\mathcal{G}^{(\sigma)}(u^*_\epsilon,p^*_\epsilon) = \mathcal{G}^{(\sigma)}(u^*,p^*)$; we can pass to the limit in (6.11) and obtain (6.28).
We have to prove that $\lim_{\epsilon\to 0}\mathcal{G}^{(\alpha)}_\epsilon(u^*_\epsilon,p^*_\epsilon) = \mathcal{G}^{(\alpha)}(u^*,p^*)$. The fact that $u^*_\epsilon\to u^*$

strongly in $L^2(0,T;V_X)$ (see Lemma 6.1) implies that $B^{(\alpha)}_{\epsilon,X}u^*_\epsilon$ converges to $B^{(\alpha)}_X u^*$ in $L^2(0,T,(V_X)')$. On the other hand, $\phi p^*_\epsilon \to \phi p^*$ weakly in $L^2(0,T;V_X)$. This implies that

$$\lim_{\epsilon\to 0}\int_0^T \left\langle B^{(\alpha)}_{\epsilon,X}u^*_\epsilon, \phi p^*_\epsilon\right\rangle = \int_0^T \left\langle B^{(\alpha)}_X u^*, \phi p^*\right\rangle.$$

It can also be checked that $(1-\phi)B^{(\alpha)}_{\epsilon,X}u^*_\epsilon$ strongly converges to $(1-\phi)B^{(\alpha)}_X u^*$ in $L^2((0,T)\times(0,X))$. From the weak convergence of p^*_ϵ to p^* in $L^2((0,T)\times(0,X))$, we deduce that

$$\lim_{\epsilon\to 0}\int_0^T\int_0^X \left((1-\phi)B^{(\alpha)}_{\epsilon,X}u^*_\epsilon\right)p^*_\epsilon = \int_0^T\int_0^X \left((1-\phi)B^{(\alpha)}_X u^*\right)p^*.$$

The two points above yield that $\lim_{\epsilon\to 0}\mathcal{G}^{(\alpha)}_\epsilon(u^*_\epsilon,p^*_\epsilon) = \mathcal{G}^{(\alpha)}(u^*,p^*)$. This and (6.12) yield (6.29). The last condition (6.30) is obtained in the same manner. ☐

6.2. Conclusion: optimality condition for the calibration problem. For calibrating the Lévy process, one observes the spot price S and the prices $(\bar p_i)_{i\in I}$ of a family of American put options with maturities/strikes given by (T_i,x_i), and we call $\bar u_i = \bar p_i - x_i + S$, $i\in I$. We assume that

$$\bar u_i > u_\circ(x_i), \qquad \text{for all } i\in I.$$

Call $T=\max_{i\in I}T_i$. Let $\bar X$ be such that for all $(\sigma,\alpha,\psi)\in\mathcal{H}$, the exercise price $\gamma(t)$ is smaller than $\bar X$ for all $t\le T$, and take $X\ge\bar X$. The calibration problem has the form (6.3) with the new definition of J:

$$J(u) = \sum_{i\in I}\omega_i(u(T_i,x_i)-\bar u_i)^2,$$

where ω_i are positive weights.
As above, we can also define the modified least square problem (6.4), and have ϵ tend to 0. Let a subsequence $(\sigma^*_{\epsilon_n},\alpha^*_{\epsilon_n},\psi^*_{\epsilon_n},u^*_{\epsilon_n})$ of solutions of (6.4) converge to $(\sigma^*,\alpha^*,\psi^*,u^*)$ as in Lemma 6.1, then $(\sigma^*,\alpha^*,\psi^*,u^*)$ is a solution of (6.3).
We assume that $u^*(T_i,x_i)>u_\circ(x_i)$, for all $i\in I$. It is clear from the continuity of u^* and from the uniform convergence of $u^*_{\epsilon_n}$ that there exists a positive real number a and an integer N such that for $n>N$, $u^*_{\epsilon_n}(t,x)>u_\circ(x)+\epsilon_n$ for all (t,x) such that $|t-T_i|<a$ and $|x-x_i|<a$ for some $i\in I$. We may fix a smooth function ϕ taking the value 1 for all x such that $|x-x_i|\ge a/2$, $|T_i-t|\ge a/2$ for all $i\in I$, and vanishing in neighborhoods of (T_i,x_i), $i\in I$.
Calling A_X the operator defined by (4.2), (3.9) and (2.12) with the parameters $(\sigma,\alpha,\psi)=(\sigma^*,\alpha^*,\psi^*)$, we obtain the optimality conditions exactly as in § 6.1.4:

THEOREM 6.6. *Under the assumptions made at the beginning of § 6.2, there exists a function $p^*\in L^2((0,T)\times(0,X))$ and a Radon measure ξ^* such that for all $v\in Z$, (Z is defined by (6.14))*

$$\int_0^T\int_0^X\left(\frac{\partial v}{\partial t}+A_Xv\right)p^* + <\xi^*,v> = 2\sum_{i\in I}\omega_i(u^*(T_i,x_i)-\bar u_i)v((T_i,x_i)), \quad (6.31)$$

and (6.17), (6.18), and such that (6.28), (6.29) and (6.30) are satisfied for all $(\sigma,\alpha,\psi)\in\mathcal{H}$, with $\mathcal{G}^{(\sigma)}$, $\mathcal{G}^{(\alpha)}$ and $\mathcal{G}^{(\psi)}$ defined respectively by (6.25), (6.26) and (6.27), (with the new choice of ϕ).

Proof. The proof follows exactly the same lines as that of Proposition 6.5. ☐

7. Appendix 1.

Proof of Lemma 3.2. By the change of variable $y = \log(x)$,

$$|v|^2_{\phi,s} = \int_{\mathbb{R}} e^y dy \int_{\mathbb{R}} \frac{\phi(z)}{|z|^{1+2s}} \left(v(e^{y-z}) - v(e^y)\right)^2 dz = \int_{\mathbb{R}} dy \int_{\mathbb{R}} \frac{\phi(z)}{|z|^{1+2s}} \left(e^{\frac{z}{2}} \tilde{v}(y-z) - \tilde{v}(y)\right)^2 dz.$$

By Fubini's theorem, $|v|^2_{\phi,s} = \int_{\mathbb{R}} dz \phi(z)|z|^{-(1+2s)} \int_{\mathbb{R}} \left(e^{\frac{z}{2}} \tilde{v}(y-z) - \tilde{v}(y)\right)^2 dy$; after a Fourier transform with respect to the variable y,

$$\begin{aligned}
|v|^2_{\phi,s} &= \int_{\mathbb{R}} dz \phi(z)|z|^{-(1+2s)} \int_{\mathbb{R}} \left| e^{z(\frac{1}{2}-i\xi)} \widehat{v}(\xi) - \widehat{v}(\xi) \right|^2 d\xi \\
&= \int_{\mathbb{R}} dz \phi(z)|z|^{-(1+2s)} \int_{\mathbb{R}} (e^z + 1 - 2e^{\frac{z}{2}} \cos(\xi z)) |\widehat{v}(\xi)|^2 d\xi \\
&= \int_{\mathbb{R}} dz \phi(z)|z|^{-(1+2s)} \int_{\mathbb{R}} ((e^{\frac{z}{2}} - 1)^2 + 4e^{\frac{z}{2}} \sin^2(\frac{\xi z}{2})) |\widehat{v}(\xi)|^2 d\xi \\
&= \left(\int_{\mathbb{R}} \frac{\phi(z)}{|z|^{1+2s}} (e^{\frac{z}{2}} - 1)^2 dz \right) \|v\|^2_{L^2(\mathbb{R}_+)} + 4 \int_{\mathbb{R}} dz \frac{\phi(z) e^{\frac{z}{2}}}{|z|^{1+2s}} \int_{\mathbb{R}} \sin^2(\frac{\xi z}{2}) |\widehat{v}(\xi)|^2 d\xi.
\end{aligned}$$

But $4 \int_{\mathbb{R}} dz \frac{\phi(z) e^{\frac{z}{2}}}{|z|^{1+2s}} \int_{\mathbb{R}} \sin^2(\frac{\xi z}{2}) |\widehat{v}(\xi)|^2 d\xi = 4 \int_{\mathbb{R}} |\xi|^{2s} |\widehat{v}(\xi)|^2 \int_{\mathbb{R}} \frac{\phi(z) e^{\frac{z}{2}}}{|\xi z|^{1+2s}} \sin^2(\frac{\xi z}{2})) |\xi| dz$. Define

$$C_1 = \int_{\mathbb{R}} \phi(z) |z|^{-(1+2s)} (e^{\frac{z}{2}} - 1)^2 dz,$$

which is a real number since $z \mapsto \phi(z) \max(e^z, 1)$ is bounded. Similarly, there exists a constant $\beta > 0$ such that

$$4 \int_{\mathbb{R}} \frac{\phi(z) e^{\frac{z}{2}}}{|\xi z|^{1+2s}} \sin^2(\frac{\xi z}{2}) |\xi| dz \le 4\beta \int_{\mathbb{R}} \frac{\sin^2(\frac{\xi z}{2})}{|\xi z|^{1+2s}} |\xi| dz = 4\beta \int_{\mathbb{R}} \frac{\sin^2(\frac{u}{2})}{|u|^{1+2s}} du.$$

and we introduce $C_2 = 4\beta \int_{\mathbb{R}} \frac{\sin^2(\frac{u}{2})}{|u|^{1+2s}} du$. We have obtained that

$$|v|^2_{\phi,s} \le C_1 \|v\|^2_{L^2(\mathbb{R}_+)} + C_2 \int_{\mathbb{R}} |\xi|^{2s} |\widehat{v}(\xi)|^2 d\xi.$$

On the other hand,

$$\int_{\mathbb{R}} \frac{\phi(z) e^{\frac{z}{2}}}{|\xi z|^{1+2s}} \sin^2(\frac{\xi z}{2}) |\xi| dz = \int_{\mathbb{R}} \phi(\frac{u}{|\xi|}) e^{\frac{u}{2|\xi|}} \frac{\sin^2(\frac{u}{2})}{|u|^{1+2s}} du \ge \int_{-1}^{1} \phi(\frac{u}{|\xi|}) e^{\frac{u}{2|\xi|}} \frac{\sin^2(\frac{u}{2})}{|u|^{1+2s}} du$$

implies that

$$\liminf_{|\xi| \to \infty} \int_{\mathbb{R}} \phi(z) e^{\frac{z}{2}} |\xi z|^{-(1+2s)} \sin^2(\frac{\xi z}{2}) |\xi| dz \ge \phi(0) \int_{-1}^{1} \sin^2(\frac{u}{2}) |u|^{-(1+2s)} du,$$

which shows that there exists a constant $M > 0$ such that for $|\xi| \ge M$,

$$4 \int_{\mathbb{R}} \phi(z) e^{\frac{z}{2}} |\xi z|^{-(1+2s)} \sin^2(\frac{\xi z}{2}) |\xi| dz \ge 2\phi(0) \int_{-1}^{1} \sin^2(\frac{u}{2}) |u|^{-(1+2s)} du,$$

and we introduce $C_4 = 2\phi(0) \int_{-1}^{1} \frac{\sin^2(\frac{u}{2})}{|u|^{1+2s}} du$. Thus

$$|v|^2_{\phi,s} \ge C_1 \|v\|^2_{L^2(\mathbb{R}_+)} + C_4 \int_{|\xi|>M} |\xi|^{2s} |\widehat{v}(\xi)|^2 d\xi \ge \frac{C_1}{2} \|v\|^2_{L^2(\mathbb{R}_+)} + C_3 \int_{\mathbb{R}} |\xi|^{2s} |\widehat{v}(\xi)|^2 d\xi,$$

with $C_3 = \min(C_1/(2M^{2s}), C_4)$. ☐

Proof of Lemma 3.3. It is enough to prove that $B|_{\mathcal{D}(\mathbb{R}_+)}$ is continuous from $\mathcal{D}(\mathbb{R}_+)$ endowed with the norm of V^s to $V^{s-\max(2\alpha,1)}$ if $\alpha \neq 1/2$, and to $V^{s-1-\epsilon}$ if $\alpha = 1/2$. For that, we use the change of variable $y = \log(x)$ and call $\tilde{u}$ the function defined on $\mathbb{R}$ by $\tilde{u}(y) = u(e^y)e^{\frac{y}{2}}$. This yields that $\langle Bu, v\rangle = \langle \widetilde{B}\tilde{u}, \tilde{v}\rangle$, where

$$\begin{aligned}\widetilde{B}\tilde{u}(y) &= -\int_{\mathbb{R}} k(z)e^z \left((1-e^{-z})(\frac{\partial \tilde{u}}{\partial y}(y) - \frac{1}{2}\tilde{u}(y)) + (e^{\frac{z}{2}}\tilde{u}(y-z) - \tilde{u}(y))\right) dz \\ &= -\int_{\mathbb{R}} k(z)e^z \left((1-e^{-z})\frac{\partial \tilde{u}}{\partial y}(y) + (\frac{1}{2}e^{-z} - \frac{3}{2})\tilde{u}(y) + e^{\frac{z}{2}}\tilde{u}(y-z)\right) dz.\end{aligned}$$

The Fourier transform of $\widetilde{B}\tilde{u}$ is

$$\hat{b} = -\widehat{\tilde{u}}(\xi)\int_{\mathbb{R}} k(z)e^z \left((1-e^{-z})i\xi + \frac{1}{2}e^{-z} - \frac{3}{2} + e^{\frac{z}{2}}e^{-i\xi z}\right) dz,$$

We make out three cases:

1) $\alpha > 1/2$. In this case, one sees that

$$\begin{aligned}(1-e^{-z})i\xi + \frac{1}{2}e^{-z} - \frac{3}{2} + e^{\frac{z}{2}}e^{-i\xi z} =& \frac{1}{2}(e^{-z} + 2e^{\frac{z}{2}} - 3) + e^{\frac{z}{2}}(e^{-i\xi z} - 1 + iz\xi) \\ & - i\xi(e^{-z} - 1 + ze^{\frac{z}{2}}).\end{aligned}$$

Therefore

$$\hat{b} = -\widehat{\tilde{u}}(\xi)\left(\begin{array}{l}\displaystyle\int_{\mathbb{R}} \frac{k(z)}{2}(1 + 2e^{\frac{3z}{2}} - 3e^z) + \int_{\mathbb{R}} k(z)e^{\frac{3z}{2}}(e^{-i\xi z} - 1 + iz\xi) \\ \displaystyle -i\xi\int_{\mathbb{R}} k(z)e^z(e^{-z} - 1 + ze^{\frac{z}{2}})\end{array}\right). \tag{7.1}$$

From the assumption on ψ, the first integral is a real number independent of ξ. As in Lemma 3.2, by introducing $\theta = \xi/|\xi|$ for $\xi \neq 0$, and writing that

$$\int_{\mathbb{R}} k(z)e^{\frac{3z}{2}}(e^{-i\xi z} - 1 + iz\xi)dz = |\xi|^{2\alpha}\int_{\mathbb{R}} |y|^{-(1+2\alpha)}\psi\left(\frac{y}{|\xi|}\right)e^{\frac{3y}{2|\xi|}}(e^{-iy\theta} - 1 + iy\theta)dy,$$

we see from the assumptions on ψ that there exists a positive constant C_1 such that $\left|\int_{\mathbb{R}} k(z)e^{\frac{3z}{2}}(e^{-i\xi z} - 1 + iz\xi)dz\right| \leq C_1|\xi|^{2\alpha}$, because $y \mapsto \psi(y)\exp(3y/2)$ is a real bounded function and $\int_{\mathbb{R}} \left|e^{-iy\theta} - 1 + iy\theta\right||y|^{-(1+2\alpha)}dy$ can be bounded independently of ξ. The third integral in (7.1) is a real number independent of ξ. Therefore,

$$|\hat{b}| \lesssim (1 + |\xi| + |\xi|^{2\alpha})|\widehat{\tilde{u}}(\xi)| \lesssim (1 + |\xi|^{2\alpha})|\widehat{\tilde{u}}(\xi)|.$$

2) $\alpha < 1/2$. We can still split $\hat{b}$ as in (7.1). From the assumption on ψ, the first integral is a real number independent of ξ. The second integral can be split into the sum of $\int_{\mathbb{R}} k(z)e^{\frac{3z}{2}}(e^{-i\xi z} - 1)dz$ and $i\xi\int_{\mathbb{R}} k(z)e^{\frac{3z}{2}}zdz$, which are both bounded by $C_1|\xi|$. The third integral is a real number independent of ξ, so

$$|\hat{b}| \lesssim (1 + |\xi|)|\widehat{\tilde{u}}(\xi)|.$$

3) $\alpha = 1/2$. The only change with respect to the previous two cases concerns the second integral: we write it as

$$|\xi|\int_{\mathbb{R}} \psi\left(\frac{y}{|\xi|}\right)e^{\frac{3y}{2|\xi|}}\frac{(e^{-iy\theta} - 1 + iy1_{|y|<1}\theta)}{|y|^2}dy + i\theta|\xi|\int_{|z\xi|>1} \psi(z)\, e^{\frac{3z}{2}}\frac{z}{|z|^2}dz. \tag{7.2}$$

The first integral in (7.2) is bounded by a constant independent of ξ, because $z \mapsto \psi(z)\exp(3z/2)$ is a bounded function and $\int_{\mathbb{R}} |y|^{-2}\left|e^{-iy\theta} - 1 + iy 1_{|y|<1}\theta\right|$ can be bounded independently of ξ. For the second integral in (7.2), we have, if $\xi > 1$

$$\left|\int_{|z\xi|>1} \psi(z)\, e^{\frac{3z}{2}} \frac{z}{|z|^2} dz\right| \le \int_{|z|>1} \frac{\psi(z)\, e^{\frac{3z}{2}}}{|z|} dz + \int_{|\xi|^{-1}\le|z|\le 1} \frac{\psi(z)\, e^{\frac{3z}{2}}}{|z|} dz \lesssim (1+\log(\xi))$$

$$|\hat{b}| \lesssim (1 + |\xi| + |\xi \log(|\xi|)|)|\widehat{u}(\xi)|. \qquad \square$$

Proof of Lemma 3.5. It is enough to prove the result for $u, v \in \mathcal{D}(\mathbb{R}_+)$:

$$\langle Bu, v\rangle = -\int_{\mathbb{R}_+} dx \int_{\mathbb{R}} k(z) \left(x(e^z - 1)\frac{\partial u}{\partial x}(x) + e^z(u(xe^{-z}) - u(x)) \right) v(x) dz = I + II + III,$$

where

$$I = \int_{\mathbb{R}_+} dx \int_{\mathbb{R}} k(z) e^z (u(x) - u(xe^{-z}))(v(x) - v(xe^{-z})) dz,$$
$$II = -\int_{\mathbb{R}_+} dx \int_{\mathbb{R}} k(z) x(e^z - 1)\frac{\partial u}{\partial x}(x) v(x) dz,$$
$$III = \int_{\mathbb{R}_+} dx \int_{\mathbb{R}} k(z) e^z (u(x) - u(xe^{-z})) v(xe^{-z}) dz.$$

But

$$II = \int_{\mathbb{R}_+} dx \int_{\mathbb{R}} k(z) x(e^z - 1)\frac{\partial v}{\partial x}(x) u(x) dz + \int_{\mathbb{R}_+} dx \int_{\mathbb{R}} k(z)(e^z - 1) u(x) v(x) dz.$$

From this,

$$\begin{aligned} II + III = &\int_{\mathbb{R}_+} \left(\int_{\mathbb{R}} k(z) \left(x(e^z - 1)\frac{\partial v}{\partial x}(x) + e^z(v(xe^{-z}) - v(x)) \right) dz \right) u(x) dx \\ &+ \left(\int_{\mathbb{R}} k(z)(2e^z - e^{2z} - 1) dz \right) \int_{\mathbb{R}_+} u(x) v(x) dx. \end{aligned}$$

The desired result is obtained. $\square$

Proof of Lemma 3.6. The assertion is already proved in the case $\alpha < 1/2$, thanks to Lemma 3.3 and Remark 4. Thus, let us focus on the case when $\alpha \ge 1/2$: after a few calculations, one sees that

$$(B^T u - Bu)(x) = \int_{\mathbb{R}} k(z) \left(x(e^z - 1)(2\frac{\partial u}{\partial x}(x) + u(x)) + e^z u(xe^{-z}) - e^{2z} u(xe^{-z}) \right) dz.$$

The same change of variables as in the proof of Lemma 3.3 leads to $\langle (B - B^T)u, v\rangle = \langle (\widetilde{B} - \widetilde{B}^T)\tilde{u}, \tilde{v}\rangle$ where

$$(\widetilde{B}\tilde{u} - \widetilde{B}^T\tilde{u})(y) = -\int_{\mathbb{R}} k(z) \left(2(e^z - 1)\frac{\partial \tilde{u}}{\partial y}(y) + e^{\frac{3z}{2}}(\tilde{u}(y - z) - \tilde{u}(y + z)) \right) dz.$$

The Fourier transform of $(\widetilde{B} - \widetilde{B}^T)\tilde{u}$ is

$$\begin{aligned} &-\widehat{\tilde{u}}(\xi) \int_{\mathbb{R}} k(z) \left(2i\xi(e^z - 1) - e^{\frac{3z}{2}}(e^{i\xi z} - e^{-i\xi z}) \right) dz \\ =& -2i\xi\widehat{\tilde{u}}(\xi) \int_{\mathbb{R}} k(z)(e^z - 1 - ze^{\frac{3z}{2}}) dz + \widehat{\tilde{u}}(\xi) \int_{\mathbb{R}} k(z) e^{\frac{3z}{2}} \left(e^{i\xi z} - e^{-i\xi z} - 2i\xi z \right) dz. \end{aligned}$$

From the assumptions, the first integral in the sum above is a real number. Let us focus on the second integral: since the function $z \mapsto e^{i\xi z} - e^{-i\xi z} - 2i\xi z$ is odd, $\psi(0)\int_{\mathbb{R}} |z|^{-(1+2\alpha)} e^{\frac{-3|z|}{2}} \left(e^{i\xi z} - e^{-i\xi z} - 2i\xi z\right) dz = 0$, and

$$\int_{\mathbb{R}} k(z) e^{\frac{3z}{2}} \left(e^{i\xi z} - e^{-i\xi z} - 2i\xi z\right) dz = \int_{\mathbb{R}} |z|^{-(1+2\alpha)} z\omega(z) \left(e^{i\xi z} - e^{-i\xi z} - 2i\xi z\right) dz$$
$$\lesssim \int_{\mathbb{R}} |z|^{-2\alpha} e^{-\zeta|z|} \left(e^{i\xi z} - e^{-i\xi z} - 2i\xi z\right) dz \lesssim |\xi|^{2\alpha-1} \lesssim (1+|\xi|),$$

where, in the case $\alpha = 1/2$, we have used the fact that $|\sin(\xi z)|/|z| \leq |\xi|$. This concludes the proof. □

8. Appendix 2.

Proof of Proposition 4.1. Consider X', $0 < X' < X$ and let ϕ be a smooth cut-off function taking the value 1 in $[0, 3/4X' + \frac{1}{4}X]$ and 0 in $[3/4X + \frac{1}{4}X', X]$. It is a possible to prove that $A_X(\mathcal{E}_X(\phi v)) \in L^2((\mathbb{R}_+))$, which yields that $\mathcal{E}_X(\phi v) \in V^2$ and $\phi v \in W_X^2$. This yields the first statement of Proposition 4.1.
Assume that $v \in V_X$ is such that $A_X v \in L^2((0,X))$. Then there exists $f \in L^2((0,X))$ such that

$$-\sigma x^2 \frac{\partial^2 v}{\partial x^2} = f - B_X v. \tag{8.1}$$

If $0 \leq \alpha < 1/2$, then, from Lemma 3.3, $B_X v \in L^2((0,X))$, and (8.1) implies that $v \in W_X^2 \cap V_X$.
If $\alpha > 1/2$, then, from Lemma 3.3, $B_X v \in V_X^{1-2\alpha}$. From this and (8.1), one immediately deduces that $v \in V_X \cap W_X^{3-2\alpha}$. A boot-strap argument is needed for improving this result:
if $1/2 < \alpha < 3/4$, then for all $\epsilon > 0$, $v \in W_X^{3/2-\epsilon}$, and $\mathcal{E}_X(v) \in V^{3/2-\epsilon}$. Note that we cannot give a better regularity result for $\mathcal{E}_X(v)$, (for example $\mathcal{E}_X(v) \in V^{3/2+\epsilon}$), because this would require the condition $\frac{\partial v}{\partial x}(x = X) = 0$, which is not proved. Then Lemma 3.3 yields that $B_X v \in L^2((0,X))$, and that $v \in W_X^2 \cap V_X$ from (8.1). In the case $\alpha = 1/2$, we obtain from Lemma 3.3 that $v \in W_X^2 \cap V_X$ as well. If $\alpha = 3/4$, the same argument shows that $v \in W_X^{2-\epsilon} \cap V_X$ for all $\epsilon > 0$.
On the contrary, if $3/4 < \alpha < 1$, we have to keep on boot-strapping: $v \in V_X \cap W_X^{3-2\alpha}$ implies that $B_X v \in V_X^{3-4\alpha}$, and from (8.1), $v \in W_X^{5-4\alpha}$. Either $3/4 < \alpha < 7/8$, and we see that there exists $\epsilon > 0$ such that $v \in W_X^{3/2+\epsilon}$, or $7/8 \leq \alpha < 1$, and we keep on boot-strapping. After a finite number of steps, we obtain the first two statements of Proposition 4.1.
Then we obtain that $\frac{\partial v}{\partial x} \in \mathcal{C}^0((0,X))$ from Sobolev imbeddings. □

Proof of Theorem 4.2. For brevity, and since the proof uses rather classical arguments, we shall omit some details. By using results on parabolic equations with monotone operators [26] page 156, it is possible to prove that (4.6) has a unique weak solution in $L^2(0,T;V_X) \cap C^0([0,T];L^2(0,X))$, with $\frac{\partial u_{X,\epsilon}}{\partial t} \in L^2(0,T;V_X')$. Note that for all t_0, $0 < t_0 < T$, $u_{X,\epsilon}$ is smooth in $(t_0,T] \times [a,b]$, where $[a,b]$ is any interval strictly contained in $(0,S)$ or in (S,X). From (3.11), the weak maximum principle may be used. It yields that, almost everywhere, $u_{X,\epsilon}$ is nonnegative on the one hand, and greater than or equal to $x \mapsto S - x$ on the other hand. Therefore, for almost every time t, $u_{X,\epsilon}(t) \in K_X$. This implies that $0 \leq rx(1 - 1_{\{x>S\}}) \leq rx(1 - 1_{\{x>S\}}\mathcal{V}_\epsilon(u_{X,\epsilon})) \leq rx$.
From this and (4.6), $u_{X,\epsilon}$ belongs to $C^0([0,T];V_X) \cap L^2(0,T;D_X)$, $\frac{\partial u_{X,\epsilon}}{\partial t} \in L^2((0,T) \times (0,X))$ and the norms $\|u_{X,\epsilon}\|_{L^\infty(0,T;V_X)}$, $\|u_{X,\epsilon}\|_{L^2(0,T;D_X)}$, $\|\frac{\partial}{\partial t} u_{X,\epsilon}\|_{L^2((0,T)\times(0,X))}$ are bounded independently of ϵ.
Since $V \subset \mathcal{C}^0((0,+\infty))$ and since for any t, $\lim_{x \to 0} u_{X,\epsilon}(t,x) = S$ (because $S - x \leq$

$u_X(t,x) \le S$), we see that $\mathcal{E}_X(u_{X,\epsilon}) \in \mathcal{C}^0([0,T]\times[0,+\infty))$.
The maximum principle yields (4.8) and (4.9). From the bounds $u_\circ(x) \le u_{X,\epsilon}(t,x) \le u^{(E)}(t,x)$, and from the fact that $\frac{\partial u^{(E)}}{\partial x}(t,0) = -1$, we see that $u_{X,\epsilon}(t,x)$ has a derivative with respect to x at $x=0$ and that $\frac{\partial u_{X,\epsilon}}{\partial x}(t,0) = -1$, $\forall t \ge 0$.
By calling $y_{X,\epsilon}$ the time derivative of $u_{X,\epsilon}$, we see that

$$\begin{aligned} &\frac{\partial y_{X,\epsilon}}{\partial t} + A_X y_{X,\epsilon} - rx1_{\{x>S\}}\mathcal{V}_\epsilon'(u_{X,\epsilon})y_{X,\epsilon} = 0,\ t\in(0,T],\ 0<x<X, \\ &y_{X,\epsilon}(t,X) = 0 \quad t\in(0,T]. \end{aligned} \tag{8.2}$$

Note that

$$-rx1_{\{x>S\}}\mathcal{V}_\epsilon'(u_{X,\epsilon}) \ge 0. \tag{8.3}$$

Since $y_{X,\epsilon} \in \mathcal{C}^0([0,T],V_X')$, we have that $y_{X,\epsilon}(t=0) = -A_X u_\circ|_{(0,X)} + rx1_{x<S} = \frac{\sigma^2 S^2}{2}\delta_{x=S} - B_X u_\circ|_{(0,X)}$. It can be seen that $-B_X u_\circ|_{(0,X)}$ is a positive distribution in V_X', because $u_\circ$ is convex: to prove it, one can approximate $u_\circ$ in V_X by a sequence $u_{\circ,n}$ of smooth convex functions with bounded support such that $-B_X u_{\circ,n} \ge 0$, and pass to the limit. Therefore, $y_{X,\epsilon}(t=0) \ge 0$ in V_X'. From this, and from (8.2, 8.3), we deduce that $y_{X,\epsilon} \ge 0$ a.e.. Therefore $u_{X,\epsilon}$ is nondecreasing w.r.t. t.
Finally, the quantities $\|u_{X,\epsilon}\|_{L^\infty(0,T;L^2(0,X))}$ and $\|u_{X,\epsilon}\|_{L^2(0,T;V_X)}$ can be bounded independently of X by taking $u_{X,\epsilon}$ as a test function in the weak formulation of (4.6) and by observing that the constants in Gårding's inequality for A_X do not depend of X. □

Proof of Theorem 4.3. We know that $u_{X,\epsilon}$ belongs to $\mathcal{C}^0([\tau,T];D_X)$ for all τ, $0<\tau<T$. Therefore, from Proposition 4.1, $u_{X,\epsilon} \in \mathcal{C}^0([\tau,T];W_X^{3/2+\epsilon})$ for some positive ϵ. This yields that for each time $t>0$, $u_{X,\epsilon} \in \mathcal{C}^1((0,X])$. On the other hand, we know that $u_{X,\epsilon}(t,X) = 0$ for $t\in[0,T]$, and $u_{X,\epsilon} \ge 0$ in $[0,T]\times[0,X]$. From the last three observations, we see that for all t, $0<t\le T$, $\frac{\partial u_{X,\epsilon}}{\partial x}(t,X) \le 0$.
We aim at proving that for each $t>0$ there exists a number $\xi(t)$, $0\le\xi(t)<X$, such that $\frac{\partial u_{X,\epsilon}}{\partial x}(t,x) \le 0$ if $\xi(t)<x<X$. Indeed, it it was not case, we would be in one of the following two cases:
1) $\frac{\partial u_{X,\epsilon}}{\partial x}(t,x) > 0$ in some interval $[y(t),X)$, $y(t)<X$. This implies that $u_{X,\epsilon}(t,x) < 0$ in $(y(t),X)$, which is impossible since $u(t,.) \ge u_\circ$.
2) There exists a strictly increasing sequence of numbers y_n, $0<y_n<y_{n+1}<X$, such that $\lim_{n\to\infty} y_n = X$ and $\frac{\partial u_{X,\epsilon}}{\partial x}(t,y_n) = 0$, $\frac{\partial u_{X,\epsilon}}{\partial x}(t,x)$ is positive for x in (y_{2n},y_{2n+1}), and negative for x in (y_{2n+1},y_{2n+2}). The numbers y_{2n}, $n\in\mathbb{N}$ are local minima of $u_{X,\epsilon}(t,.)$. Let us consider the terms entering equation (4.6) at $x=y_{2n}$: we have $\frac{\partial u_{X,\epsilon}}{\partial t}(t,y_{2n}) \ge 0$ and $\lim_{n\to\infty}\frac{\partial u_{X,\epsilon}}{\partial t}(t,y_{2n}) = 0$. It is clear that $-\frac{\sigma^2 y_{2n}^2}{2}\frac{\partial^2 u_{X,\epsilon}}{\partial x^2}(t,y_{2n}) \le 0$ and $ry_{2n}\frac{\partial u_{X,\epsilon}}{\partial x}(t,y_{2n}) = 0$ because y_{2n} is a local minimum. We also know that $ry_{2n}(1-1_{\{y_{2n}>S\}}\mathcal{V}_\epsilon(u_{X,\epsilon}(t,y_{2n}))) \ge 0$ and that $\lim_{n\to\infty} ry_{2n}(1-1_{\{y_{2n}>S\}}\mathcal{V}_\epsilon(u_{X,\epsilon}(t,y_{2n}))) = 0$. Therefore

$$\liminf_{n\to\infty} B_X u_{X,\epsilon}(t,y_{2n}) \ge 0,$$

and, since $\frac{\partial u_{X,\epsilon}}{\partial x}(t,y_{2n}) = 0$,

$$\limsup_{n\to\infty}\int_{\mathbb{R}} k(z)\left(e^z(1_{z>\log\frac{y_{2n}}{X}}u_{X,\epsilon}(t,y_{2n}e^{-z}) - u_{X,\epsilon}(t,y_{2n}))\right)dz \le 0.$$

This yields $\int_{z>0} k(z)e^z u_{X,\epsilon}(t,Xe^{-z})dz \le 0$, which is impossible since $u_{X,\epsilon} \ge u_\circ$.
Therefore, for all $t>0$, the function $x\mapsto\frac{\partial u_{X,\epsilon}}{\partial x}(t,x)$ is nonpositive in a neighborhood of X, and $(\frac{\partial u_{X,\epsilon}}{\partial x}(t,.))_+$ is zero near X.

Moreover, since $u_{X,\epsilon}$ is nondecreasing and $u_{X,\epsilon}(\cdot,X)=0$, the function $t\mapsto \frac{\partial u_{X,\epsilon}}{\partial x}(t,X)$ is nonincreasing. Therefore, there exists τ_0, $0\le\tau_0\le T$, such that $\frac{\partial u_{X,\epsilon}}{\partial x}(t,X)<0$ for $\tau_0<t\le T$ and $\frac{\partial u_{X,\epsilon}}{\partial x}(t,X)=0$ for $0\le t\le\tau_0$.
By taking the derivative of (4.6) with respect to x and multiplying by x, we see that $z_{X,\epsilon}=x\frac{\partial u_{X,\epsilon}}{\partial x}$ satisfies

$$\begin{aligned}
&\frac{\partial z_{X,\epsilon}}{\partial t}+A_X z_{X,\epsilon}-rx1_{\{x>S\}}\mathcal{V}'_\epsilon(u_{X,\epsilon})z_{X,\epsilon} \\
&=-rx(1-1_{x>S}\mathcal{V}_\epsilon(u_{X,\epsilon}))+rS\mathcal{V}_\epsilon(u_{X,\epsilon})\delta_{x=S}, \qquad t\in(0,T],\ 0<x<X, \\
&z_{X,\epsilon}(t=0,x)=-x1_{0<x<S}, \qquad 0<x<X.
\end{aligned} \tag{8.4}$$

Since $z_{X,\epsilon}(t,X)=0$ for $t\in[0,\tau_0]$, (τ_0 is defined above), the function $z_{X,\epsilon}|_{t\in(0,\tau_0)}\in L^2(0,\tau_0;V_X)$. On the other hand, $z_{X,\epsilon}(t,.)\notin V_X$, for $t\in(\tau_0,T]$. In (8.4), for $t>\tau_0$, $A_X z_{X,\epsilon}(t)$ i.e.

$$\begin{aligned}
A_X z_{X,\epsilon}(t,x)=-\frac{\sigma^2x^2}{2}\frac{\partial^2 z_{X,\epsilon}}{\partial x^2}(t,x)+rx\frac{\partial z_{X,\epsilon}}{\partial x}(t,x) \\
-\int_{\mathbb{R}}k(z)\left(x(e^z-1)\frac{\partial z_{X,\epsilon}}{\partial x}(t,x)+e^z(1_{\{z>-\log(X/x)\}}z_{X,\epsilon}(t,xe^{-z})-z_{X,\epsilon}(t,x))\right)dz,
\end{aligned}$$

has a sense as a distribution and for all $X'<X$, belongs to the dual of $\{v\in V_X, v=0 \text{ in } (X',X)\}$.
We split the function $z_{X,\epsilon}$ into the sum of two functions $\tilde z_{X,\epsilon}\in\mathcal{C}^0([0,T];L^2(0,X))$ and $\hat z_{X,\epsilon}\in L^2(0,T;V_X)$ which satisfy

$$\begin{aligned}
&\frac{\partial \hat z_{X,\epsilon}}{\partial t}+A_X\hat z_{X,\epsilon}-rx1_{\{x>S\}}\mathcal{V}'_\epsilon(u_{X,\epsilon})\hat z_{X,\epsilon}=rS\mathcal{V}_\epsilon(u_{X,\epsilon})\delta_{x=S}, \\
&\hat z_{X,\epsilon}(t=0,x)=0, \qquad 0<x<X, \\
&\hat z_{X,\epsilon}(t,X)=0, \qquad 0<t<T,
\end{aligned} \tag{8.5}$$

and

$$\begin{aligned}
&\frac{\partial \tilde z_{X,\epsilon}}{\partial t}+A_X\tilde z_{X,\epsilon}-rx1_{\{x>S\}}\mathcal{V}'_\epsilon(u_{X,\epsilon})\tilde z_{X,\epsilon}=-rx(1-1_{x>S}\mathcal{V}_\epsilon(u_{X,\epsilon})), \\
&\tilde z_{X,\epsilon}(t=0,x)=-x1_{0<x<S}, \qquad 0<x<X, \\
&\tilde z_{X,\epsilon}(t,X)\le 0, \qquad 0<t<T.
\end{aligned} \tag{8.6}$$

From the fact that $u_{X,\epsilon}\ge \underline{u}_X^{(E)}$ and from (4.7), we know that

$$\lim_{\epsilon\to0}\|\mathcal{V}_\epsilon(u_{X,\epsilon}(S))\delta_{x=S}\|_{L^2(0,T;V')}=0.$$

Thus, $\lim_{\epsilon\to0}\|\hat z_{X,\epsilon}\|_{L^2(0,T;V)}=0$. One can also prove that $\hat z_{X,\epsilon}(t,0)=0$ and that $\hat z_{X,\epsilon}\ge0$.
From the last observation and since $z_{X,\epsilon}(t,0)=0$, we see that for all $t\in[0,T]$, $\tilde z_{X,\epsilon}(t,0)=0$.
We know that $\tilde z_{X,\epsilon}|_{(0,\tau_0)}\in L^2(0,\tau_0,V_X)$: in $(0,\tau_0)\times(0,X)$, we can take $e^{-Mt}(\tilde z_{X,\epsilon})_+$ as a test-function in the equation satisfied by $\tilde z_{X,\epsilon}$. From Gårding's inequality, choosing M large enough yields that $\tilde z_{X,\epsilon}(t,\cdot))_+=0$ for $t\in[0,\tau_0]$.
On the other hand, for $\tau_1>\tau_0$, there exists a constant $\underline{z}>0$ such that $\tilde z_{X,\epsilon}(t,X)\le-\underline{z}$ for $t\in[\tau_1,T]$. This and the continuity of $\tilde z_{X,\epsilon}$, imply that there exits $\underline{X}_{\tau_1}$, $S<\underline{X}_{\tau_1}<X$, such that $\tilde z_{X,\epsilon}\le0$ in $[\tau_1,T]\times[\underline{X}_{\tau_1},X]$. Therefore, in the time interval $[\tau_1,T]$, we can take $(\tilde z_{X,\epsilon}(t,x))_+e^{-Mt}$ as a test-function in the equation satisfied by $\tilde z_{X,\epsilon}$, even if $\tilde z_{X,\epsilon}$ does not belong to V_X, (indeed $(\tilde z_{X,\epsilon}(t,.))_+$ does not see the singular behavior of $z_{X,\epsilon}(t,.)$ near X). From Gårding's inequality, we have that for M large enough, $t\mapsto e^{-Mt}\int_0^X((\tilde z_{X,\epsilon})_+)^2(t,x)dx$ is nonincreasing in (τ_1,T).

We can have τ_1 tend to τ_0. This yields that $(\tilde{z}_{X,\epsilon})_+ = 0$ in $(\tau_0, T) \times (0, X)$. We have proved that $\tilde{z}_{X,\epsilon} \leq 0$ in $(0,T)\times(0,X)$.
Finally, let X and X' be two numbers such that $S < X < X'$. Call $\tilde{u}_{X,\epsilon}$ the function obtained by extending $u_{X,\epsilon}$ by 0 in $[0,T]\times[X,X']$. Clearly, $\tilde{u}_{X,\epsilon} \in C^0([0,T];K_{X'})$. It can be seen from (4.6) and from $\frac{\partial u_{X,\epsilon}}{\partial x}(x = X) \leq 0$ that $\frac{\partial \tilde{u}_{X,\epsilon}}{\partial t} + A_{X'}\tilde{u}_{X,\epsilon} + rx(1 - 1_{\{x>S\}}\mathcal{V}_\epsilon(\tilde{u}_{X,\epsilon}))$ is a negative distribution in $(0,T)\times(0,X')$. This and the maximum principle imply (4.10). □

Proof of Proposition 4.8. It is enough to prove that $\mu_{X,\epsilon}$ converges to μ_X in $L^1((0,T)\times(0,X))$ because $0 \leq \mu_{X,\epsilon} \leq rx$. For that, we make two observations:
a)Since $u_{X,\epsilon}$ is nondecreasing w.r.t. t, $\mu_{X,\epsilon}$ is nonincreasing w.r.t. t.
b)

$$\frac{\partial \mu_{X,\epsilon}}{\partial x} = r1_{x>S}\mathcal{V}_\epsilon(u_{X,\epsilon}) + rS\mathcal{V}_\epsilon(u_{S,\epsilon})\delta_{x=S} + r1_{x>S}\mathcal{V}'_\epsilon(u_{X,\epsilon}))\tilde{z}_{X,\epsilon} + r1_{x>S}\mathcal{V}'_\epsilon(u_{X,\epsilon}))\hat{z}_{X,\epsilon},$$

where $\tilde{z}_{X,\epsilon}$ and $\hat{z}_{X,\epsilon}$ are respectively defined in (8.6) and (8.5). The first three terms in the right hand side are positive distributions. Let us study more carefully the last one: we call $g_\epsilon = r1_{\{x>S\}}\mathcal{V}'_\epsilon(u_{X,\epsilon})\hat{z}_{X,\epsilon}$. We know that $\hat{z}_{X,\epsilon}$ is nonnegative and tends to 0 in $L^2(0,T;V_X)$. Hence, g_ϵ is a nonpositive function. Moreover, let ϕ_η be a smooth function defined on $[0,X]$ such that $0 \leq \phi_\eta \leq 1$, $\phi_\eta = 1$ for $0 \leq x \leq X-\eta$, and $\phi_\eta(x) = 0$ for $X - \eta/2 \leq x \leq X$. Taking ϕ_η as a test function in (8.5) yields

$$\lim_{\epsilon\to 0}\left(\int_0^X \hat{z}_{X,\epsilon}(T,x)\phi_\eta(x)dx + \int_0^T \langle A_X\hat{z}_{X,\epsilon}, \phi_\eta\rangle - \int_0^T\int_0^X g_\epsilon(t,x)\phi_\eta(x)dxdt\right) = 0.$$

This proves that $\lim_{\epsilon\to 0}\|g_\epsilon\|_{L^1((0,T)\times(0,X-\eta))} = 0$.
To summarize, $\mu_{X,\epsilon}|_{\{x<X-\eta\}}$ is the sum of a nondecreasing function and of $\tilde{\mu}_{X,\epsilon} = \int_0^x g_\epsilon(t,y)dy$, and $\tilde{\mu}_{X,\epsilon}$ and its derivative w.r.t. x tend to 0 in $L^1((0,T)\times(0,X-\eta))$. From a) and b), one sees that the total variation of $\mu_{X,\epsilon}$ on $(0,T)\times(0,X-\eta)$ is bounded. Therefore, we can extract a subsequence of $\mu_{X,\epsilon}|_{\{x<X-\eta\}}$ converging strongly in $L^1((0,T)\times(0,X-\eta))$. The limit cannot be anything but $\mu_X|_{\{x<X-\eta\}}$, so the whole sequence converges to $\mu_X|_{\{x<X-\eta\}}$. Since η is arbitrarily small and $\mu_{X,\epsilon}$ is bounded, we have that $\lim_{\epsilon\to 0}\|\mu_{X,\epsilon} - \mu_X\|_{L^1((0,T)\times(0,X))} = 0$.
The convergence results for $u_{X,\epsilon}$ are an easy consequence of the strong convergence of $\mu_{X,\epsilon}$ to μ_X.□

Proof of Lemma 6.2. The proof is similar to an argument given in [2]. For brevity, we shall omit some details. We call $\mathcal{Q}_\epsilon$ the bilinear form on $L^2((0,T)\times(0,X))\times Z_\epsilon$:

$$\mathcal{Q}_\epsilon(q,v) = \int_0^T\int_0^X \left(\frac{\partial v}{\partial t} + A_{\epsilon,X}v - rx1_{\{x>S\}}\mathcal{V}'_\epsilon(u^*_\epsilon)v\right)q.$$

It is clear that $\mathcal{Q}_\epsilon$ is continuous. Moreover, there exists a positive constant c, independent of ϵ, such that

$$\inf_{q\in L^2((0,T)\times(0,X))}\sup_{v\in Z_\epsilon}\frac{\mathcal{Q}_\epsilon(q,v)}{\|q\|_{L^2((0,T)\times(0,X))}\|v\|_{Z_\epsilon}} \geq c.$$

To prove this inf-sup condition, take $v \in L^2(0,T;V_X)\cap H^1(0,T;L^2((0,X)))$ as the weak solution of

$$\frac{\partial v}{\partial t} + A_{\epsilon,X}v - rx1_{\{x>S\}}\mathcal{V}'_\epsilon(u^*_\epsilon)v = q \quad t>0, \qquad v(0,\cdot) = 0.$$

and observe that $\|v\|_{Z_\epsilon} \leq C\|q\|_{L^2((0,T)\times(0,X))}$ for a constant C independent of ϵ.
Therefore, calling Q_ϵ the linear and continuous operator from $L^2((0,T)\times(0,X))$

to the dual of Z_ϵ defined by $< Q_\epsilon p, v >= \mathcal{Q}_\epsilon(p,v)$, the range of Q_ϵ is closed and Q_ϵ is injective. On the other hand, $\mathcal{Q}_\epsilon(q,v)$ for all $q \in L^2((0,T)\times(0,X))$ implies that $v=0$. Therefore, Q_ϵ^T is injective. We have proved that Q_ϵ is an isomorphism from $L^2((0,T)\times(0,X))$ onto the dual of Z_ϵ and that its inverse is continuous with a norm independent of ϵ.
From this and since $z \mapsto 2(u_\epsilon^*(T,x_{ob})-\bar{u})z((T,x_{ob}))$ is a continuous linear form on Z_ϵ with a continuity constant independent of ϵ, there exists a unique $p_\epsilon^* \in L^2((0,T)\times(0,X))$ such that for all $v\in Z_\epsilon$, $\mathcal{Q}_\epsilon(p_\epsilon^*,v) = 2(u_\epsilon^*(T,x_{ob})-\bar{u})v((T,x_{ob}))$ and $\|p_\epsilon^*\|_{L^2((0,T)\times(0,X))}$ is bounded independently of ϵ. The first part of the lemma is proved.
To prove the second part of the lemma, consider $G_\epsilon \in L^2((0,T)\times\mathbb{R}_+)$ the solution of the backward Cauchy problem:

$$\begin{aligned}&\frac{\partial G_\epsilon}{\partial t} + \frac{(\sigma_\epsilon^*)^2}{2}\frac{\partial^2}{\partial x^2}(x^2 G_\epsilon) - B_\epsilon^T G_\epsilon + \frac{\partial}{\partial x}(rxG_\epsilon) = 0, \qquad (t,x)\in[0,T)\times\mathbb{R}_+,\\ &G_\epsilon(t=T) = -2(u_\epsilon^*(T,x_{ob})-\bar{u})\delta_{x=x_{ob}},\end{aligned} \tag{8.7}$$

where B_ϵ^T is given by (3.6). One can check that G_ϵ is smooth for $t<T$ and that for any integer k and for any compact ω in $[0,T]\times[0,+\infty)$ which does not contain (T,x_{ob}), the norm of G_ϵ in $\mathcal{C}^k(\omega)$ is bounded independently of ϵ. Also, for the function ϕ defined in Lemma 6.2, $\phi G_\epsilon \in L^2(0,T;V)$, with a norm bounded by a constant independent of ϵ.
Let χ be a smooth function with a compact support contained in $(0,T]\times[0,X)$, taking the value 1 in a neighborhood of (T,x_{ob}), and whose support does not intersect the support of $\mathcal{V}_\epsilon(u_\epsilon^*)$ for all ϵ. For example, $\chi = 1-\phi$ can be chosen. With $A_{\epsilon,X}^T$ defined in Remark 11, it may be checked that $\|\chi A_\epsilon^T G_\epsilon - A_{\epsilon,X}^T(\chi G_\epsilon)\|_{L^2((0,T)\times(0,X))}$ is bounded independently of ϵ. The reason for that is that χ is constant near the point where G_ϵ is singular.
One sees that $q_\epsilon^* = p_\epsilon^* - \chi G_\epsilon$ is the unique solution (in the very weak sense defined above, i.e. by duality with the functions in Z^ϵ) of a boundary value problem in $(0,T]\times(0,X)$, of the form

$$\begin{aligned}\frac{\partial q_\epsilon^*}{\partial t} - A_{\epsilon,X}^T q_\epsilon^* + rx1_{\{x>S\}}\mathcal{V}_\epsilon'(u_\epsilon^*)q_\epsilon^* &= g_\epsilon^*, && (t,x)\in[0,T)\times(0,X),\\ q_\epsilon^*(t,X) &= 0, && t\in(0,T),\\ q_\epsilon^*(T,x) &= 0, && x\in(0,X),\end{aligned}$$

where $g^* = \frac{\partial\chi}{\partial t}G_\epsilon + \chi A_\epsilon^T G_\epsilon - A_{\epsilon,X}^T(\chi G_\epsilon) \in L^2((0,T)\times(0,X))$. This last boundary value problem has a unique weak solution in $L^2(0,T;V_X)$, with a norm bounded independently of ϵ. The weak and the very weak solutions coincide. Therefore $p_\epsilon^* - \chi G_\epsilon \in L^2(0,T;V_X)$ and $\|p_\epsilon^* - \chi G_\epsilon\|_{L^2(0,T;V_X)}$ is bounded by a constant independent of ϵ. Therefore $\phi p_\epsilon^* \in L^2(0,T;V_X)$, with a norm bounded independently of ϵ. □

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