

Course Notes (Paris 2009)

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Abstract

The following are notes on stochastic and parametric PDEs of the short course in Paris.

1 Lecture 4: Capturing Functions in Infinite Dimensions

Finally, we want to give an example where the problem is to recover a function of infinitely many variables. We will first show how such problems occur in the context of stochastic partial differential equations.

1.1 Elliptic equations: general principles

We consider the elliptic equation

$$-\nabla \cdot (a \nabla u) = f \quad \text{in } D, \quad u|_{\partial D} = 0, \quad (1.1)$$

in a bounded Lipschitz domain $D \subset \mathbb{R}^d$, where $f \in L_2(D)$. There is a rich theory for existence and uniqueness for equations of this form which we briefly recall.

Central to the theory of elliptic equations is the Sobolev space $V := H_0^1(D)$ (called the energy space) which is a Hilbert space equipped with the energy norm $\|v\|_V := \|\nabla v\|_{L^2(D)}$. The dual of V is $V^* = H^{-1}(D)$. The solution of (1.1) is defined in weak form as a function $u \in H_0^1(D)$ which satisfies

$$\int_D a(x) \nabla u(x) \cdot \nabla v(x) dx = \int_D f(x) v(x) dx, \quad \text{for all } v \in H_0^1(D), \quad (1.2)$$

where the gradient ∇ is taken with respect to the x variable. This formulation shows that the Lax-Milgram theory applies. In particular, a sufficient condition for the existence and uniqueness of the solution u is that a satisfies the ellipticity assumption

$$0 < a_{\min} \leq a(x) \leq a_{\max}, \quad x \in D. \quad (1.3)$$

Under this assumption, the solution satisfies the estimate

$$\|u\|_V \leq C_0 := \frac{\|f\|_{V^*}}{a_{\min}}. \quad (1.4)$$

The same theory applies even if a is complex valued. Now the lower ellipticity condition replaces a by $\operatorname{Re}(a)$ in (1.3) and the upper condition is that $|a|$ is uniformly bounded.

There is also a general principal of perturbation for elliptic equations which shows to some extent the smooth dependence of the solution on the diffusion coefficient a . If a, \tilde{a} are two such coefficients with the ellipticity constants $a_{\min}, \tilde{a}_{\min} \geq \mu > 0$, then the solutions u and \tilde{u} with identical right side f will satisfy

$$\|u - \tilde{u}\|_{H_0^1(D)} \leq C(\mu) \|a - \tilde{a}\|_{L^\infty(D)}. \quad (1.5)$$

2 Stochastic equations

We are interested in the case of stochastic equations where $a = a(x, \omega)$ is now a real valued random field on some probability space (Ω, Σ, P) but f remains a deterministic function. The solution $u = u(x, \omega)$ is now a random field associated to the same probability space. Stochasticity describes the uncertainty in the diffusion coefficient a . In order to ensure uniform ellipticity, one assumes

Assumption S: *There exist constants $0 < a_{\min} \leq a_{\max}$ such that*

$$a_{\min} \leq a(x, \omega) \leq a_{\max}, \quad (x, \omega) \in D \times \Omega. \quad (2.1)$$

There are two general numerical approaches to stochastic elliptic PDEs: Monte-Carlo (MC) methods and deterministic methods.

Monte-Carlo (MC) methods: These methods approximate quantities such as the mean ($\bar{u}(x) := \mathbb{E}(u(x)) = \int_{\Omega} u(x, \omega) dP(\omega)$) or higher moments of u . One takes N independent draws of a and computes the solution $u_i, i = 1, \dots, N$, corresponding to each of these draws and then uses the u_i to estimate the quantities of interest. For example, the average $\bar{u}_N := \frac{1}{N} \sum_{i=1}^N u_i$ gives an estimate in expectation

$$\mathbb{E}(\|\bar{u} - \bar{u}_N\|_V) \leq (\mathbb{E}(\|u\|_V^2))^{1/2} N^{-\frac{1}{2}} \quad (2.2)$$

i.e. Monte-Carlo approximations with N samples converge with rate $1/2$ provided that the solution u as a V -valued random function has finite second moments. Unfortunately, the rate $N^{-1/2}$ cannot be improved for MC.

In practice, the u_i are computed approximately by space discretization, for example by the finite element method. But we will leave this issue aside in this talk and instead focus on whether other (deterministic) methods could potentially outperform Monte-Carlo. Our benchmark is N which is the number of times we need to solve a corresponding elliptic equation.

Deterministic methods: These have been studied for several decades. In contrast to MC, these methods take advantage of the smooth dependence of u on a . We will consider the *spectral approach* which is based on the so-called Wiener generalized polynomial chaos expansion. The first step consists in representing a by a sequence of scalar random variables $(y_j)_{j \geq 1}$, usually

obtained through a decomposition of the oscillation $a - \bar{a}$ into an orthogonal basis $(\psi_j)_{j \geq 1}$ of $L_2(D)$:

$$a(x, \omega) = \bar{a}(x) + \sum_{j \geq 1} y_j(\omega) \psi_j(x). \quad (2.3)$$

Here, the reader can think of $\{\psi_j\}$ as his favorite basis, for example a wavelet basis or Fourier basis.

The solution is now viewed as a function $u(x, y)$ where $x \in D$ is the space variable and $y = (y_j)_{j \geq 1}$ is a vector of “stochastic variables”, and the objective is to compute a numerical approximation to $u(x, y)$. Any such approximation would give us access to all information about the solution u . Note that

$$y_j := \|\psi_j\|_{L_2(D)}^{-2} \int_D (a - \bar{a}) \psi_j, \quad j = 1, 2, \dots \quad (2.4)$$

Of course, for each draw $\omega \in \Omega$, y is just a sequence of real numbers. So in the end we can consider parametric problems for real sequences y .

Up to a renormalization of the basis functions ψ_j , we may assume without loss of generality that for all $j \geq 1$ the random variables y_j are such that $\|y_j\|_{L_\infty(\Omega)} = 1$. Up to a change of the definition of a on a set of measure zero in Ω this is equivalent to

$$\sup_{\omega \in \Omega} |y_j(\omega)| = 1. \quad (2.5)$$

The vector y is thus a point in the infinite dimensional cube

$$U := [-1, 1]^{\mathbb{N}},$$

i.e. the unit ball of $\ell^\infty(\mathbb{N})$. Hence in going further, we assume that the ψ_j has been so normalized.

3 Parametric elliptic equations

Now, we make a major but illuminating change in our point of view. Rather than view the vectors y of interest as only those that arise as realizations of the stochastic process, we instead admit any $y \in U$ as being viable. For any such $y \in U$, we define

$$a(x, y) = \bar{a}(x) + \sum_{j \geq 1} y_j \psi_j(x). \quad (3.1)$$

and further define $u(x, y)$ as the solution to the (parametric) elliptic equation

$$-\nabla_x(a(x, y) \nabla_x u(x, y)) = f(x), \quad x \in D, \quad (3.2)$$

with boundary condition

$$u(x, y) = 0, \quad x \in \partial D.$$

In order to guarantee uniform ellipticity for $(x, y) \in D \times U$, we use

Assumption P: We assume that there are constants a_{\min} and a_{\max} such that

$$0 < a_{\min} \leq a(x, y) \leq a_{\max} < +\infty, \quad (x, y) \in D \times U.$$

This assumption is very close to the assumption imposed in the stochastic setting and in fact implies the stochastic ellipticity assumption. The only difference in these two conditions is that the set of y for which we require ellipticity may be larger in **Assumption P**.

Remark: Notice that solving the parametric problem (3.2) will certainly give the solution to the stochastic problem. However, we may be solving (3.2) for values of y which do not arise from draws of $\omega \in \Omega$.

4 Compressible representations of u

We would like to efficiently capture the function $u(x, y)$ for all $(x, y) \in D \times U$. This would in turn allow us to solve the stochastic problem as well. The goal of our work with Albert Cohen and Chris Schwab is to show that under very mild conditions on a , the solution $u(x, y)$ can be efficiently represented by a polynomial expansion in y with coefficients in V . To describe this, we introduce the standard multivariate notation. We let \mathcal{F} be the set of all sequences $\nu = (\nu_1, \nu_2, \dots)$ such that ν has finite support and each entry in ν is a nonnegative integer. So $|\nu| = \sum_{j \geq 1} \nu_j$ is always finite. If $\alpha = (\alpha_j)_{j \geq 1}$ is a sequence of positive numbers, we define for all $\nu \in \mathcal{F}$

$$\alpha^\nu := \prod_{j \geq 1} \alpha_j^{\nu_j}.$$

We also use the following sequence b throughout

$$b = (b_j)_{j=1}^\infty, \quad b_j := \frac{\|\psi_j\|_{L^\infty(D)}}{a_{\min}}. \quad (4.1)$$

The theorems that follow which guarantee a sparse representation of $u(x, y)$ will assume some decay for the sequence (b_j) . Let us observe that such decay conditions follow from very moderate assumptions on the smoothness of a .

Remark: Recall that the normalization of ψ_j is determined by the stochastic problem and the fact that we recast it in the parameter domain U . It is easy to see that even very minimal smoothness conditions on $a(x, \omega)$ will result in decay of (b_j) . Let us show how this goes only in the simple case of a one-dimensional Fourier expansion,

$$a(x, \omega) = \bar{a}(x) + \sum_{k \in \mathbb{Z}} \hat{a}(k, \omega) e^{i2\pi kx}.$$

It is known that if the function $a(\cdot, \omega) - \bar{a}$ is in $\text{Lip}(s, L^1)$ for some $s > 1$, then its Fourier coefficients satisfy the decay estimate

$$|a(k, \omega)| \leq C|k|^{-s}, \quad |k| \geq 1,$$

with C depending on the $\text{Lip}(s, L^1)$ -norm of $a(\cdot, \omega) - \bar{a}$. Assuming that this norm is bounded independently of ω and returning to (2.4) gives that

$$\|\psi_j\|_{L^\infty(D)} \leq Cj^{-s}, \quad j = 1, 2, \dots \quad (4.2)$$

when we reindex the basis. Therefore ℓ^p summability of the sequence $(b_j)_{j \geq 1}$ is ensured when $s > \frac{1}{p}$.

We now know that mild smoothness conditions on a translate into decay conditions on the (b_j) so that a has a compressible decomposition on U :

$$a(x, y) = \bar{a}(x) + \sum_{j \geq 1} y_j \psi_j(x), \quad (4.3)$$

where $(\|\psi_j\|_{L^\infty(D)})$ is in ℓ_p . However, we are not so much interested in approximating a which we know but rather the solution $u(x, y)$. We are therefore interested in seeing where these decay conditions on (b_j) translate into compressible representation of $u(x, y)$. This is indeed the case.

In [1], we showed the following theorem.

Theorem 4.1 *If (i) $\sum_{j \geq 1} b_j < 1$, and (ii) $(b_j) \in \ell_p$ for some $p < 1$, then*

$$u(x, y) = \sum_{\nu \in \mathcal{F}} c_\nu(x) y^\nu, \quad (4.4)$$

where the functions $c_\nu(x)$ are in V and $(\|c_\nu\|_V) \in \ell_p$ for the same value of p .

Remark: *This theorem shows that the compressibility of a translates into the same compressibility of u .*

Let us say a few words about the proof of this theorem before giving it a hard looking over. To prove the theorem, we need to give estimates for $\|c_\nu\|_V$. This is not too hard. For a fixed $y \in U$, we know that for all $v \in V$

$$\int_D a(x, y) \nabla u(x, y) \nabla v(x) dx = \int_D f(x) v(x) dx.$$

Differentiating this identity with respect to the variable y_j gives

$$\int_D a(x, y) \nabla \partial_{y_j} u(x, y) \nabla v(x) dx + \int_D \psi_j(x) \nabla u(x, y) \nabla v(x) dx = 0. \quad (4.5)$$

We claim that more generally for every $v \in V$ holds

$$\int_D a(x, y) \nabla \partial_y^\nu u(x, y) \nabla v(x) dx + \sum_{\{j: \nu_j \neq 0\}} \nu_j \int_D \psi_j(x) \nabla \partial_y^{\nu - e_j} u(x, y) \nabla v(x) dx = 0, \quad (4.6)$$

where e_j is the Kronecker sequence with value 1 at position j and 0 elsewhere. (4.6) is proved by induction on $|\nu|$ using the same idea as used in deriving (4.5). From (4.6) it is not difficult to prove

$$\|\partial_y^\nu u(\cdot, y)\|_V \leq C_0 \sum_{\{j: \nu_j \neq 0\}} \nu_j b_j (|\nu| - 1)! b^{\nu - e_j} = C_0 \left(\sum_{\{j: \nu_j \neq 0\}} \nu_j \right) (|\nu| - 1)! b^\nu = C_0 |\nu|! b^\nu, \quad \nu \in \mathcal{F}.$$

One now proves the representation (4.4) with $c_\nu(x) := \frac{D^\nu u(x,0)}{\nu!}$ (see [1] for details).

While there is certainly a beauty in the above theorem, we are not here to praise it but rather to point out its deficiencies. This centers around the relevance of the assumption (i) (we have already argued (ii) is very reasonable). The motivation for (i) lies in the ellipticity assumption that must be imposed for $a(x, y)$. From this condition, we know that

$$\bar{a}(x) + \sum_{j \geq 1} y_j \psi_j(x) \geq \bar{a}(x) - \sum_{j \geq 1} \|\psi_j\|_{L_\infty(D)} \geq a_{\min} - \sum_{j \geq 1} \|\psi_j\|_{L_\infty(D)} > 0. \quad (4.7)$$

In other words, we do get the ellipticity condition we want. However, if the functions ψ_j are not global, this condition is much too strong and ellipticity can be guaranteed by the potentially much weaker condition

$$\sum_{j \geq 1} |\psi_j(x)| \leq \bar{a}(x) - a_{\min}, \quad x \in D. \quad (4.8)$$

which is actually equivalent to the lower inequality in **Assumption P**. Moving to this weaker condition is important in applications where we use bases such as a wavelet basis whose power lies in part in the fact that they are local. It therefore becomes an important question as to whether the above theorem can be generalized to hold under the weaker assumption (4.8). We shall indeed show this is the case but to do so we need to make another conceptual step and consider parametric problems with complex parameters.

5 Complex parametric problems

Let us recall that the problem before us is to hopefully prove Theorem 4.1 under the weaker **Assumption P** whose lower inequality is equivalent to (4.8). We shall do this by considering the extension of the parametric problem to complex parameters.

We want to go from the real variables y to the complex variables z and so we now define $a(x, z)$ as in (3.1) but with z now complex. If we are on any domain \tilde{U} of complex sequences $z = (z_1, \dots, z_n, \dots)$ for which the series (3.1) defining $a(x, z)$ converges and satisfies

$$0 < r \leq \operatorname{Re}(a(x, z)) \leq |a(x, z)| \leq R < \infty, \quad (5.1)$$

then the solution $u(x, z)$ to the elliptic equation with diffusion coefficient $a(x, z)$ is unique and satisfies

$$\sup_{z \in \tilde{U}} \|u(\cdot, z)\|_V \leq \frac{\|f\|_{V^*}}{r}. \quad (5.2)$$

Let us see that **Assumption P** allows us to conclude the validity of (5.1) for analytic domains that are significantly larger than U . Indeed, let us take $r := a_{\min}/2$ and take any sequence $\rho = (\rho_j)_{j \geq 1}$ where $\rho_j > 0$ and $\max(\rho_j, 1) = 1 + \epsilon_j$ with

$$\sum_{j \geq 1} \epsilon_j |\psi_j(x)| \leq r. \quad (5.3)$$

Then **Assumption P** (see (4.8)) shows that (5.1) will be satisfied for any $z = (z_j)$ provided that

$$|\operatorname{Re}(z_j)| \leq \rho_j, \quad |z_j| \leq R, \quad j \geq 1$$

for some constant $R > 0$. In particular, if $\epsilon_j = 0$ for $j > J$, where J is arbitrary but fixed, then we obtain uniform ellipticity on the polydisk

$$U_\rho := \otimes_{j \geq 1} \{|z_j| \leq \rho_j\}, \quad \rho := (\rho_j)_{j \geq 1}. \quad (5.4)$$

5.1 Bounds for $\|c_\nu\|_V$ under Assumption P

As advertised, we shall work under the weaker condition **Assumption P** and prove that Theorem 4.1 remains valid. That is, we have the following

Theorem 5.1 *Suppose $a(x, y)$ satisfies the **Assumption P** and in addition $(b_j) \in \ell_p$, for some $p < 1$. Then,*

$$u(x, y) = \sum_{\nu \in \mathcal{F}} c_\nu(x) y^\nu, \quad y \in U, \quad (5.5)$$

where $(\|c_\nu\|_V)_{\nu \in \mathcal{F}} \in \ell_p$ for this same value of p .

Sketch of Proof:

1. Fix any $J \geq 1$ and define $u_J(z_1, \dots, z_J) := u(z_1, \dots, z_J, 0, 0, \dots)$ which is a V valued function of J complex variables. Fix any such ρ of the above form for which $\rho_j = 1$, $j > J$. We know that we have uniform ellipticity on the polydisc U_ρ (see (5.3) and (5.4)). This gives that $\|u_J(x, z)\|_V \leq \frac{\|f\|_{V^*}}{r}$ uniformly for $z \in U_\rho$. The next step is to show that u_J is holomorphic (i.e. infinitely differentiable in U_ρ). This is proved in the same manner that we have estimated $\|\partial_y^\nu u(\cdot, y)\|_V$ using the real method.

2. From the holomorphy above, we have that $u_J(x, z)$ is a V -valued analytic function in the polydisc U_ρ . Let \mathcal{F}_J denote the sequences in \mathcal{F} which are supported on $\{1, \dots, J\}$. The Cauchy formula gives that for any $\nu \in \mathcal{F}_J$, we have for $c_\nu(x) := \frac{\partial_y^\nu u(x, 0)}{\nu!}$, that

$$\|c_\nu\|_V \leq C_r \prod_{j>0} \rho_j^{-\nu_j} = C_r \rho^{-\nu}, \quad (5.6)$$

with the convention that $\rho_j^{-\nu_j} = 1$ if $\nu_j = 0$.

3. An important remark is that we have at our disposal the ability to choose the sequence ρ (as long as it satisfies (5.3)). We will use this option to choose for each ν a sequence tailored to ν . Namely, we first choose J_0 large enough such that

$$\sum_{j>J_0} b_j \leq 1/16. \quad (5.7)$$

Such a J_0 exists because $b \in \ell_p \subset \ell_1$. Note that we may always assume, up to some reindexing, that the sequence b is non-increasing. We split \mathbb{N} into the two sets

$$E := \{0 < j \leq J_0\} \quad \text{and} \quad F := \{j > J_0\}.$$

For $j \in E$, we set

$$\rho_j := \kappa > 1,$$

with κ chosen so that

$$(\kappa - 1) \sum_{j \leq J_0} \|\psi_j\|_{L^\infty(D)} \leq \frac{a_{\min}}{4}. \quad (5.8)$$

For $j \in F$ we set

$$\rho_j := \frac{\nu_j}{4|\nu_F|b_j},$$

where we use the notation ν_F for the restriction of ν to a set F and $|\cdot|$ denotes the ℓ^1 norm so that $|\nu_F| := \sum_{j > J_0} \nu_j$. With such choices, we have for all $x \in D$, and for the sequence $\epsilon_j := \max(\rho_j, 1) - 1$,

$$\begin{aligned} \sum_{j > 0} \epsilon_j |\psi_j(x)| &\leq (\kappa - 1) \sum_{j \leq J_0} |\psi_j(x)| + \sum_{j > J_0} |\rho_j| |\psi_j(x)| \\ &\leq \frac{a_{\min}}{4} + \sum_{j > J_0} \frac{\nu_j a_{\min} |\psi_j(x)|}{4|\nu_F| \|\psi_j\|_{L^\infty(D)}} \\ &\leq \frac{a_{\min}}{4} + \frac{a_{\min}}{4} = \frac{a_{\min}}{2} = r \end{aligned} \quad (5.9)$$

which proves that (5.3) holds and $u_J(x, z)$ is analytic on U_ρ . Introducing the notation

$$\eta := \frac{1}{\kappa} < 1 \quad \text{and} \quad d_j := 4b_j$$

we have from (5.6)

$$\|c_\nu\|_V \leq C_r \left(\prod_{j \in E} \eta^{\nu_j} \right) \left(\prod_{j \in F} \left(\frac{|\nu_F| d_j}{\nu_j} \right)^{\nu_j} \right). \quad (5.10)$$

This is the bound we want for the $\|c_\nu\|_V$ from which we can derive the theorem.

4. Let us now discuss how one proves that the bound (5.10) implies $(\|c_\nu\|)_{\nu \in \mathcal{F}}$ is in ℓ_p . We denote by $d = (d_j)_{j \in F}$ the sequence of the d_j indexed only by the $j \in F$. Note that by assumption (5.7), we have

$$\|d\|_{\ell_1} = \sum_{j > J_0} d_j \leq \frac{1}{4}. \quad (5.11)$$

The estimate (5.10) has the general form

$$\|c_\nu\|_V \leq C_r \alpha(\nu_E) \beta(\nu_F). \quad (5.12)$$

Such a general form allows to perform a factorization in the estimate the ℓ_p norms of the $\|c_\nu\|_V$: introducing \mathcal{F}_E and \mathcal{F}_F the sequences of positive integers with finite support indexed by E and F respectively, we can write

$$\begin{aligned} \sum_{\nu \in \mathcal{F}} \|c_\nu\|_V^p &\leq C_r^p \sum_{\nu \in \mathcal{F}} \alpha(\nu_E)^p \beta(\nu_F)^p \\ &= C_r^p \left(\sum_{\nu \in \mathcal{F}_E} \alpha(\nu)^p \right) \left(\sum_{\nu \in \mathcal{F}_F} \beta(\nu)^p \right) \\ &:= C_r^p A_E A_F \end{aligned}$$

In our particular setting, the first factor A_E is easily estimated, again by factorization since we have

$$\begin{aligned} A_E &= \sum_{\nu \in \mathcal{F}_E} \alpha(\nu)^p \\ &= \sum_{\nu \in \mathcal{F}_E} \prod_{j \in E} \eta^{\nu_j p} \\ &= \prod_{j \in E} \left(\sum_{n \geq 0} \eta^{np} \right) \end{aligned}$$

and therefore

$$A_E \leq \left(\frac{1}{1 - \eta^p} \right)^{J_0} < +\infty. \quad (5.13)$$

The second factor A_F requires a bit more work because of the appearance of the factorial factors. This bound is given in the appendix.

5. Finally, we need to show that $u(x, y) = \sum_{\nu \in \mathcal{F}} c_\nu(x) y^\nu$. The analysis that has just been given shows that $u_J(x, y) = \sum_{\nu \in \mathcal{F}_J} c_\nu(x) y^\nu$. We arrive at the representation of u from the fact that $\sup_{y \in U} \|u(\cdot, y) - u_J(\cdot, y)\|_V \rightarrow 0, J \rightarrow \infty$ because of the stability result (1.5).

6 Concluding remarks

6.1 Approximation of u

If a satisfies **Assumption P** and in addition the sequence $(b_j) \in \ell_p$ for some $p < 1$, then we have proven that $(\|c_\nu\|_V)_{\nu \in \mathcal{F}}$ is also in ℓ_p and

$$u(x, y) = \sum_{\nu \in \mathcal{F}} c_\nu(x) y^\nu. \quad (6.1)$$

Hence there is a polynomial space \mathcal{P}_{Λ_N} spanned by $z^\nu, \nu \in \Lambda_N$, with $\#(\Lambda_N) \leq N$, such that

$$\|u(x, y) - \sum_{\nu \in \Lambda_N} c_\nu(x) y^\nu\|_V \leq CN^{1-1/p}. \quad (6.2)$$

The sets Λ_N can be taken to have a lot of structure. For example, they can be taken as nested as N increases. Also, one can prove that for a given Λ_N , whenever $\nu \in \Lambda_N$ then all $\mu \leq \nu$ can be taken in Λ_N .

The fact that $u(x, y)$ can be well approximated by certain polynomials (with coefficients in V) can be thought of as a regularity theorem on the structure of the solution.

6.2 Numerical algorithms

We have not discussed any numerical algorithms based on the above results. If we know the corresponding space of polynomials then algorithms can be developed based on space discretization for example by using finite element methods (see [1]).

6.3 The role of the underlying probability measure:

The main point of the above analysis is to show that under very mild smoothness conditions on $a(x, \omega)$, the solution to all of the stochastic problems that arise have a very sparse representation. This was done by moving to a parametric setting. One could ask where the original probability measure has gone? The influence of this measure is felt in the assumption that the renormalized bases $\|\psi_j\|_{L^\infty(D)}$ have a mildly fast decay. This assumption will hold if the $a(x, \omega)$ all have mild smoothness but this smoothness is required uniformly over the draws $\omega \in \Omega$. It is clear that such a theory could be developed with the weaker assumption that only with high probability the $a(x, \omega)$ have the required smoothness.

Once one has the required smoothness, the role of the probability measure is not felt since we actually show there is a polynomial space of dimension N such that for any ω the solution $u(x, \omega)$ can be approximated by V valued polynomials to accuracy $O(N^{1-1/p})$ in the V norm.

6.4 Comparison with Monte Carlo

We have begun with a discussion of the disadvantages of Monte Carlo. Have we overcome these? Well, Monte Carlo applies in a different setting where one receives N independent draws of a . For our theory to have an impact, we would have to be able to ask directed questions. It is reasonable to expect that asking N directed questions we should be able to recover $u(x, \omega)$ to accuracy $O(N^{1-1/p})$ in the V norm, uniformly in $\omega \in \Omega$.

7 Appendix: Completion of bound for A_F

In our particular setting for $\nu \in \mathcal{F}$, we have

$$\beta(\nu) := \prod_{j \in F} \left(\frac{|\nu_F| d_j}{\nu_j} \right)^{\nu_j} = \frac{|\nu_F|^{|\nu_F|}}{\prod_{j \in F} \nu_j^{\nu_j}} d^{\nu_F}, \quad (7.1)$$

where we have used the notation $d^{\nu_F} = \prod_{j \in F} d_j^{\nu_j}$ and the convention that $0^0 = 1$. We first transform the quantities of the form n^n into $n!$ by using Stirling type estimates: for all $n > 0$, we have

$$\frac{n! e^n}{e \sqrt{n}} \leq n^n \leq \frac{n! e^n}{\sqrt{2\pi} \sqrt{n}}. \quad (7.2)$$

The right hand side is actually equivalent to n^n as $n \rightarrow +\infty$, and the left hand side is easy to prove by passing to the logarithm. In addition, according to our convention

$$0^0 = 0! = 1. \quad (7.3)$$

Using the right inequality in (7.2) without even using the factor \sqrt{n} , we bound by above the quantity $|\nu_F|^{|\nu_F|}$ in (7.1) by

$$|\nu_F|^{|\nu_F|} \leq \frac{|\nu_F|! e^{|\nu_F|}}{\sqrt{2\pi}}.$$

On the other hand, using the left inequality in (7.2) as well as (7.3), we bound by below the quantity $\prod_{j \in F} \nu_j^{\nu_j}$ in (7.1) by

$$\prod_{j \in F} \nu_j^{\nu_j} \geq \frac{\nu_F! e^{|\nu_F|}}{\prod_{j \in F} \max\{1, e \sqrt{\nu_j}\}}.$$

Injecting these estimates in (7.1), this allows to bound $\beta(\nu)$ as follows

$$\beta(\nu) \leq \frac{|\nu_F|!}{\nu_F!} d^{\nu_F} \prod_{j \in F} \max\{1, e \sqrt{\nu_j}\}. \quad (7.4)$$

We now use a lemma from [1], that says that since $\|d\|_{\ell^1} = \frac{1}{4} < 1$ and since $\|d\|_{\ell^p} < +\infty$, we may factorize d as

$$d_j = \gamma_j \delta_j,$$

where γ and δ are positive sequences such that

$$\|\gamma\|_{\ell^1} < 1, \quad \|\delta\|_{\ell^\infty} < 1 \quad \text{and} \quad \|\delta\|_{\ell^q} < \infty, \quad \text{with} \quad q := \frac{p}{1-p}.$$

Then, using this factorization in (7.4) and Hölder's inequality, we obtain

$$A_F = \sum_{\nu \in \mathcal{F}_F} \beta(\nu)^p \leq \sum_{\nu \in \mathcal{F}_F} \left(\frac{|\nu_F|!}{\nu_F!} d^{\nu_F} \prod_{j \in F} \max\{1, e\sqrt{\nu_j}\} \right)^p \quad (7.5)$$

$$\leq \left(\sum_{\nu \in \mathcal{F}_F} \frac{|\nu_F|!}{\nu_F!} \gamma^\nu \right)^p \left(\sum_{\nu \in \mathcal{F}_F} \prod_{j \in F} \max\{1, (e\sqrt{\nu_j})^q\} \delta_j^{q\nu_j} \right)^{1-p} \quad (7.6)$$

We know that the first factor is bounded since $\|\gamma\|_{\ell^1} < 1$, and we actually have

$$\sum_{\nu \in \mathcal{F}_F} \frac{|\nu_F|!}{\nu_F!} \gamma^\nu = \frac{1}{1 - \|\gamma\|_{\ell^1}}.$$

The second factor can be computed by factorization

$$\sum_{\nu \in \mathcal{F}_F} \prod_{j \in F} \max\{1, (e\sqrt{\nu_j})^q\} \delta_j^{q\nu_j} = \prod_{j \in F} \left(\sum_{n \geq 0} \max\{1, (e\sqrt{n})^q\} \delta_j^{qn} \right)$$

Since $\|\delta\|_{\ell^\infty} < 1$, it is not very difficult to prove that the sums in $n \geq 0$ that appear in the product all converge and that one can bound them by

$$\sum_{n \geq 0} \max\{1, (e\sqrt{n})^q\} \delta_j^{qn} \leq 1 + C \delta_j^q,$$

where the constant C is independent of j . Since $\delta \in \ell^q$, this second factor is also bounded. This concludes the sketch of the proof of the theorem.

References

- [1] A. Cohen, R. DeVore and C. Schwab, *Convergence rates of best N -term Galerkin approximations for a class of elliptic sPDEs*, preprint.