Refined Sobolev inequalities in Lorentz spaces

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Abstract

We establish refined Sobolev inequalities between the Lorentz spaces and homogeneous Besov spaces. The sharpness of these inequalities is illustrated on several examples, in particular based on non-uniformly oscillating functions known as chirps. These results are also used to derive refined Hardy inequalities.

1 Introduction

The Sobolev inequality states that for a function f defined on $\mathbb{R}^d$,

$$\|f\|_{L^p} \leq C\|f\|_{\dot{H}^s},$$

with $\frac{1}{p} = \frac{1}{2} - \frac{s}{d}$. A refined version of this inequality was proved in [9] :

$$\|f\|_{L^p} \leq C\|f\|_{\dot{H}^s}^{1-\frac{2s}{d}} \|f\|_{\dot{B}^{s-\frac{d}{2}}_{\infty,\infty}}^{\frac{2s}{d}},$$

where $\dot{B}^{s-\frac{d}{2}}_{\infty,\infty}$ is the homogeneous Besov space. Similar results also hold when the smoothness of the function is measured in L^q, namely

$$\|f\|_{L^p} \leq C\|f\|_{\dot{B}^s_{q,q}}^{1-\frac{qs}{d}} \|f\|_{\dot{B}^{s-\frac{d}{q}}_{\infty,\infty}}^{\frac{qs}{d}}, \tag{1.1}$$

with $\frac{1}{p} = \frac{1}{q} - \frac{s}{d}$, or the same kind of inequalities with $\dot{W}^{s,q}$ in place of $\dot{B}^s_{q,q}$. We also refer to [3, 4, 12, 13, 15] for similar results, involving in particular the space BV.

These inequalities are generalizations of the usual Sobolev embedding in Lebesgue spaces, their additional feature being that they are invariant under oscillations and fractal transforms see [1]. A significant application is the description of defect of compactness in Sobolev embedding (see [8] in the L^2 framework and [10] in L^p framework).

On the other hand, it is also known that Sobolev and Besov spaces also embed in Lorentz spaces $L^{p,q}(\mathbb{R}^d)$: with $\frac{1}{p} = \frac{1}{q} - \frac{s}{d}$, one has

$$\|f\|_{L^{p,q}} \leq C\|f\|_{\dot{B}^s_{q,q}}. \tag{1.2}$$

This may be derived from the standard inequalities in L^p spaces by a real interpolation argument. This justifies looking for refined inequalities of the type (1.1) with $L^{p,q}(\mathbb{R}^d)$ in place of $L^p(\mathbb{R}^d)$.

In this paper, such inequalities will be established. The "price to pay" will be a slight modification in one index of the Besov space of negative smoothness, namely

$$\|f\|_{L^{p,q}} \leq C\|f\|_{\dot{B}^s_{q,q}}^{1-\frac{qs}{d}} \|f\|_{\dot{B}^{s-\frac{d}{q}}_{\infty,q}}^{\frac{qs}{d}}. \tag{1.3}$$

It will be shown that this change is unavoidable and that such inequalities are sharp.

The structure of the paper is the following. First, in §2, we briefly recall some facts on Lorentz and Besov spaces, we establish estimates for Besov spaces which are used in the sequel. In §3 we give the proof of the refined Sobolev inequalities (1.3), which is in the same line of idea as [9] or [12]. The sharpness of the inequality is discussed in §4. Similar improved Sobolev inequality, these inequalities remain sharp for oscillatory functions, but more interestingly the necessity of the above mentionned change of index can be proved by considering non-uniformly oscillating functions known as *chirps*. Such functions have been deeply studied by Meyer and Jaffard, see [11, 15] and arise naturally in signal processing. Finally, as a by-product of (1.3) we obtain in §5 a refined version of the Hardy inequalities in the L^q framework, namely

$$\Big(\int \frac{|f(x)|^q}{|x|^{sq}}\Big)^{\frac{1}{q}} \le C\|f\|_{\dot{B}^s_{q,q}}^{1-\frac{qs}{d}} \|f\|_{\dot{B}^{s-\frac{d}{q}}_{\infty,q}}^{\frac{qs}{d}}. \tag{1.4}$$

A similar inequality was derived in [1] in the case $q = 2$. However, our method of proof is significantly simpler and applies to arbitrary values of q. Let us stress that Hardy inequalities are of important use in analysis: among other applications, we can mention blow-up methods or the study of pseudo-differentional operators with singular coefficients.

2 Lorentz spaces and Besov spaces

For any measurable function f on $\mathbb{R}^d$, we define its distribution and rearrangement functions

$$\mu_f(t) := |\{x\ ;\ |f(x)| \ge t\} \ \text{ and } \ f^*(s) := \inf\{t\ ;\ \mu_f(t) \le s\}.$$

For $1 \le p < \infty$ and $1 \le q \le \infty$, the Lorentz spaces $L^{p,q}(\mathbb{R}^d)$ is defined as the set of all measurable functions f such that the function $s^{\frac{1}{p}} f^*(s)$ belongs to $L^q(\mathbb{R}_+, \frac{ds}{s})$. The $L^{p,q}$ norm is defined by

$$\|f\|_{L^{p,q}} := \Big(\int_0^{+\infty} (s^{\frac{1}{p}} f^*(s))^q \frac{ds}{s}\Big)^{\frac{1}{q}}, \tag{2.5}$$

with the standard modification when $q = \infty$ which corresponds to the weak L^p space $wL^p = L^{p,\infty}$. It is well known that $L^{p,p} = L^p$ and that Lorentz spaces can be derived from L^p spaces by the real interpolation method. In particular, when $1 < p < \infty$ we have $L^{p,q} = [L^1, L^\infty]_{\theta,q}$ with $\frac{1}{p} = 1 - \theta$. We refer to [5, 2] for these classical results.

In this paper, we shall use the expression of the $L^{p,q}$ norm in terms of the distribution function μ_f, namely

$$\|f\|_{L^{p,q}} = \Big(p \int_0^{+\infty} (t\mu_f(t)^{\frac{1}{p}})^q \frac{dt}{t}\Big)^{\frac{1}{q}}. \tag{2.6}$$

To derive this expression, one first writes

$$f^*(s)^q = q\int_0^{f^*(s)} t^{q-1}dt,$$

so that

$$\begin{aligned} \|f\|^q_{L^{p,q}} &= \textstyle\int_0^{+\infty} s^{\frac{q}{p}-1} f^*(s)^q ds \\ &= q\textstyle\int_0^{+\infty} s^{\frac{q}{p}-1}[\int_0^{f^*(s)} t^{q-1}dt]ds \\ &= q\textstyle\int_0^{+\infty} t^{q-1}[\int_0^{\mu_{f^*}(t)} s^{\frac{q}{p}-1}ds]dt \\ &= p\textstyle\int_0^{+\infty} t^{q-1}\mu_{f^*}(t)^{\frac{q}{p}}dt. \end{aligned}$$

Observing that $\mu_f = \mu_{f^*}$ we obtain (2.6).

There exists several equivalent definitions of Besov spaces either based on Littlewood-Paley decompositions, moduli of smoothness, approximation procedures, wavelet decompositions. We refer to [17] and [7] for a general treatment, and only recall here the definition based on Littlewood-Paley theory.

Let φ be a function of d variables such that its Fourier transform $\hat{\varphi}$ is non-negative, C^∞, supported in the ball $\{|\omega| \le M\}$ for some $M > 1$ and such that $\hat{\varphi}(\omega) = 1$ if $|\omega| \le 1$. For $j \in \mathbb{Z}$ we define the operator S_j acting on tempered distributions of d variables by

$$S_j f = 2^{dj}\varphi(2^j\cdot) * f$$

and $\Delta_j = S_{j+1} - S_j$ or equivalently

$$\Delta_j f = 2^{dj}\psi(2^j\cdot) * f,$$

with $\hat{\psi}(\omega) = \hat{\varphi}(\frac{\omega}{2}) - \hat{\varphi}(\omega)$. We define the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^d)$ as the space of tempered distribution f such that $\sum_{j=-J}^{J} \Delta_j f$ converges towards f in $\mathcal{S}'$ as $J \to +\infty$ and such that

$$\|f\|_{\dot{B}^s_{p,q}} := \|(2^{sj}\|\Delta_j f\|_{L^p})_{j\in\mathbb{Z}}\|_{\ell^q}$$

is finite. It is known and not difficult to check that this definition is independent of the initial choice of the function φ with the above mentionned property, the resulting norms being equivalent.

We shall consider Besov spaces of regularity index s which is either positive or negative. Using discrete Hardy inequalities, it is also possible to characterize these spaces using only the operator S_j : for $s > 0$, we have

$$\|f\|_{\dot{B}^s_{p,q}} \sim \|(2^{sj}\|(I - S_j)f\|_{L^p})_{j\in\mathbb{Z}}\|_{\ell^q},$$

and for $s < 0$,

$$\|f\|_{\dot{B}^s_{p,q}} \sim \|(2^{sj}\|S_j f\|_{L^p})_{j\in\mathbb{Z}}\|_{\ell^q}.$$

Equivalent norms for Besov spaces can also be obtained with the discrete frequencies 2^j replaced by a continuous one: introducing for any $\lambda > 0$ the operators

$$S_\lambda f = \lambda^d\varphi(\lambda\cdot) * f \quad\text{and}\quad \Delta_\lambda f = S_{2\lambda} f - S_\lambda f = \lambda^d\psi(\lambda\cdot) * f,$$

we then find that $\|f\|_{\dot{B}^s_{p,q}}$ is equivalent to the norm of the function

$$F(\lambda) := \lambda^s\|\Delta_\lambda f\|_{L^p} \tag{2.7}$$

in $L^q(\mathbb{R}_+, \frac{\mathrm{dt}}{\mathrm{t}})$. By similar arguments as when working with discrete frequency, we can equivalently choose

$$F(\lambda) := \lambda^s\|(I - S_\lambda)f\|_{L^p} \tag{2.8}$$

for $s > 0$ or

$$F(\lambda) := \lambda^s\|S_\lambda f\|_{L^p}, \tag{2.9}$$

for $s < 0$. For the purpose of our analysis, we shall also need estimates on the derivative $F'(\lambda) = \frac{dF}{d\lambda}$, as expressed by the following result.

Lemma 2.1 *There exists a constant K which depends only on s, d, p, q and the choice of φ such that for all $f \in \dot{B}^s_{p,q}$, the function $G(\lambda) := \lambda F'(\lambda)$ belongs to $L^q(\mathbb{R}_+, \frac{\mathrm{dt}}{\mathrm{t}})$ with*

$$\|G\|_{L^q(\mathbb{R}_+, \frac{\mathrm{dt}}{\mathrm{t}})} \le K\|f\|_{\dot{B}^s_{p,q}}, \tag{2.10}$$

where F is defined by (2.7)*, or by* (2.8) *when $s > 0$, or by* (2.9) *when $s < 0$.*

Proof: we only consider the case where F is defined by (2.7), the two others being treated in the same way. We first estimate the derivative of the function $g(\lambda) := \|\Delta_\lambda f\|_{L^p}$. Clearly, we have

$$|g'(\lambda)| \le \|\frac{d}{d\lambda}(\lambda^d\psi(\lambda\cdot)) * f\|_{L^p} = \|\lambda^{d-1}\tilde{\psi}(\lambda\cdot) * f + d\lambda^{d-1}\psi(\lambda\cdot) * f\|_{L^p},$$

with $\tilde{\psi}(x) = x\cdot\nabla\psi(x)$. Since we have

$$F'(\lambda) = s\lambda^{s-1}g(\lambda) + \lambda^s g'(\lambda),$$

it follows that

$$|G(\lambda)| \le (s + d)F(\lambda) + \lambda^s\|\lambda^d\tilde{\psi}(\lambda\cdot) * f\|_{L^p} := (s + d)F(\lambda) + \widetilde{F}(\lambda)$$

The function $\tilde{\psi}$ satisfies $\widehat{\tilde{\psi}}(\omega) = \widehat{\tilde{\varphi}}(\frac{\omega}{2}) - \widehat{\tilde{\varphi}}(\omega)$ with

$$\widehat{\tilde{\varphi}}(\omega) := \hat{\varphi}(\omega) + \omega\cdot\nabla\hat{\varphi}(\omega).$$

Since $\hat{\tilde{\varphi}}$ is C^∞, compactly supported in the ball $\{|\omega| \le M\}$, and such that $\hat{\varphi}(\omega) = 1$ if $|\omega| \le 1$, we obtain that from the definition of Besov spaces that

$$\|G\|_{L^q(\mathbb{R}_+, \frac{\mathrm{dt}}{\mathrm{t}})} \le (s + d)\|F\|_{L^q(\mathbb{R}_+, \frac{\mathrm{dt}}{\mathrm{t}})} + \|\widetilde{F}\|_{L^q(\mathbb{R}_+, \frac{\mathrm{dt}}{\mathrm{t}})} \le K\|f\|_{\dot{B}^s_{p,q}},$$

for some K which depends on (s, d, p, q, φ). ⋄

3 Proof of the refined inequality

In order to prove (1.3), we decompose f into low and high frequencies according to

$$f = S_\lambda f + (I - S_\lambda) f$$

where S_λ is the convolution operator defined in the previous section. From the characterization of Besov spaces which was recalled in §2.2, we have

$$\int_0^\infty [F(\lambda)]^q \frac{d\lambda}{\lambda} \leq C_0 \|f\|^q_{\dot{B}^{s-\frac{d}{q}}_{\infty,q}}$$

where C_0 is an absolute constant and

$$F(\lambda) := \lambda^{s-\frac{d}{q}} \|S_\lambda f\|_{L^\infty}.$$

We also have

$$\int_0^{+\infty} \lambda^{qs} \|(I - S_\lambda) f\|^q_{L^q} \frac{d\lambda}{\lambda} \leq C_1 \|f\|^q_{\dot{B}^s_{q,q}},$$

where C_1 is also an absolute constant.

From triangle inequality we have

$$\{x \; ; \; |f(x)| > t\} \subset \{x \; ; \; |S_\lambda f(x)| > \frac{t}{2}\} \cup \{x \; ; \; |(I - S_\lambda) f(x)| > \frac{t}{2}\}$$

We now consider the particular value

$$t = t(\lambda) = 2\lambda^{\frac{d}{p}} F(\lambda).$$

For such a value, we have $|S_\lambda f(x)| \leq \frac{t(\lambda)}{2}$ for all x, so that

$$\{x \; ; \; |f(x)| > t(\lambda)\} \subset \{x \; ; \; |(I - S_\lambda) f| > \frac{t(\lambda)}{2}\},$$

i.e. $\mu_f(t(\lambda)) \leq \mu_{(I-S_\lambda)f}(\frac{t(\lambda)}{2})$. Using this observation, we can estimate the $L^{p,q}$ norm of f according to

$$\begin{aligned}
\|f\|^q_{L^{p,q}} &= p \int_0^{+\infty} t^q \mu_f(t)^{\frac{q}{p}} \frac{dt}{t} \\
&\leq p \int_0^{+\infty} t(\lambda)^{q-1} t'(\lambda) [\mu_{(I-S_\lambda)f}(\tfrac{t(\lambda)}{2})]^{\frac{q}{p}} d\lambda \\
&= 2^q d \int_0^{+\infty} [\lambda^{\frac{d}{p}} F(\lambda)]^{q-1} (\lambda^{\frac{d}{p}-1} F(\lambda)) [\mu_{(I-S_\lambda)f}(\tfrac{t(\lambda)}{2})]^{\frac{q}{p}} d\lambda \\
&\quad + 2^q p \int_0^{+\infty} [\lambda^{\frac{d}{p}} F(\lambda)]^{q-1} (\lambda^{\frac{d}{p}} F'(\lambda)) [\mu_{(I-S_\lambda)f}(\tfrac{t(\lambda)}{2})]^{\frac{q}{p}} d\lambda \\
&= 2^q d \int_0^{+\infty} \lambda^{\frac{dq}{p}-1} F(\lambda)^q [\mu_{(I-S_\lambda)f}(\tfrac{t(\lambda)}{2})]^{\frac{q}{p}} d\lambda \\
&\quad + 2^q p \int_0^{+\infty} \lambda^{\frac{dq}{p}} F(\lambda)^{q-1} F'(\lambda) [\mu_{(I-S_\lambda)f}(\tfrac{t(\lambda)}{2})]^{\frac{q}{p}} d\lambda \\
&:= T_0 + T_1
\end{aligned}$$

In order to estimate further the two above terms, we remark that

$$\mu_{(I-S_\lambda)f}(\frac{t(\lambda)}{2}) \leq (\frac{t(\lambda)}{2})^{-q} \|(I - S_\lambda) f\|^q_{L^q} = (\lambda^{\frac{d}{p}} F(\lambda))^{-q} \|(I - S_\lambda) f\|^q_{L^q}.$$

For T_0, this gives

$$\begin{aligned}
T_0 &\leq 2^q d \int_0^{+\infty} \lambda^{\frac{dq}{p}-1} F(\lambda)^q [(\lambda^{\frac{d}{p}} F(\lambda))^{-q} \|(I - S_\lambda) f\|^q_{L^q}]^{\frac{q}{p}} d\lambda \\
&= 2^q d \int_0^{+\infty} F(\lambda)^{q(1-\frac{q}{p})} [\lambda^{d-\frac{dq}{p}} \|(I - S_\lambda) f\|^q_{L^q}]^{\frac{q}{p}} \frac{d\lambda}{\lambda} \\
&= 2^q d \int_0^{+\infty} F(\lambda)^{q(1-\frac{q}{p})} [\lambda^{qs} \|(I - S_\lambda) f\|^q_{L^q}]^{\frac{q}{p}} \frac{d\lambda}{\lambda}.
\end{aligned}$$

Applying Hölder inequality, we thus obtain

$$\begin{aligned}
T_0 &\leq 2^q d \Big(\int_0^{+\infty} F(\lambda)^q \frac{d\lambda}{\lambda}\Big)^{1-\frac{q}{p}} \Big(\int_0^{+\infty} \lambda^{qs} \|(I - S_\lambda) f\|^q_{L^q} \frac{d\lambda}{\lambda}\Big)^{\frac{q}{p}} \\
&\leq 2^q d \Big(\int_0^{+\infty} F(\lambda)^q \frac{d\lambda}{\lambda}\Big)^{\frac{qs}{d}} \Big(\int_0^{+\infty} \lambda^{qs} \|(I - S_\lambda) f\|^q_{L^q} \frac{d\lambda}{\lambda}\Big)^{1-\frac{qs}{d}} \\
&\leq C\Big(\|f\|^{\frac{qs}{d}}_{\dot{B}^{s-\frac{d}{q}}_{\infty,q}} \|f\|^{1-\frac{qs}{d}}_{\dot{B}^s_{q,q}}\Big)^q.
\end{aligned}$$

where C is an absolute constant. Proceeding in an exactly similar manner for T_1, we obtain

$$T_1 \leq 2^q p\Big(\int_0^{+\infty} F(\lambda)^{q-1}\lambda F'(\lambda)\frac{d\lambda}{\lambda}\Big)^{\frac{qs}{d}}\Big(\int_0^{+\infty} \lambda^{qs}\|(I-S_\lambda)f\|_{L^q}^q\frac{d\lambda}{\lambda}\Big)^{1-\frac{qs}{d}}.$$

Using Lemma 2.1 together with a Hölder inequality, we see that the integral in the first factor can again be controlled by $\|f\|^q_{\dot{B}^{s-\frac{d}{q}}_{\infty,q}}$, and therefore we also obtain

$$T_1 \leq C\Big(\|f\|^{\frac{qs}{d}}_{\dot{B}^{s-\frac{d}{q}}_{\infty,q}}\|f\|^{1-\frac{qs}{d}}_{\dot{B}^s_{q,q}}\Big)^q,$$

where C is an absolute constant. Combining our estimates for T_0 and T_1 we have proved that

$$\|f\|^q_{L^{p,q}} \leq C\Big(\|f\|^{\frac{qs}{d}}_{\dot{B}^{s-\frac{d}{q}}_{\infty,q}}\|f\|^{1-\frac{qs}{d}}_{\dot{B}^s_{q,q}}\Big)^q,$$

which is the desired result.

4 Sharpness of the inequality

We shall to discuss the sharpness of (1.3) through two examples. The first one shows that the refined estimates are sharp with respect to oscillating functions: consider for all positive ω, the function

$$\varphi_\omega(x) = e^{i\omega x_1}\varphi(x),$$

where φ is a fixed function in $\mathcal{S}(\mathbb{R}^d)$. For such a function, $\|\varphi_\omega\|_{L^{p,q}}$ is independent of ω since the $L^{p,q}$ norm only depends on the modulus. On the other hand

$$\|\varphi_\omega\|_{\dot{B}^s_{q,q}} \sim \omega^s.$$

Therefore, the Sobolev inequality (1.2) is not sharp when ω is large. It is also easily checked that

$$\|\varphi_\omega\|_{\dot{B}^{s-\frac{d}{q}}_{\infty,q}} \sim \omega^{s-\frac{d}{q}},$$

and therefore

$$\|\varphi_\omega\|^{1-\frac{qs}{d}}_{\dot{B}^s_{q,q}}\|\varphi_\omega\|^{\frac{qs}{d}}_{\dot{B}^{s-d/q}_{\infty,q}} \sim \omega^{s(1-\frac{qs}{d})+(s-\frac{d}{q})\frac{qs}{d}} = 1,$$

which shows that (1.3) is sharp for such oscillatory functions.

The defect of this first example is that it does not make much distinction between (1.3) and the improved Sobolev inequality (1.1) in L^p spaces : indeed, we also have that $\|\varphi_\omega\|_{L^p}$ is independent of ω and that $\|\varphi_\omega\|_{\dot{B}^{s-\frac{d}{q}}_{\infty,\infty}} \sim \omega^{s-\frac{d}{q}}$. In particular, this example does not reveal the necessity of using $\dot{B}^{s-\frac{d}{q}}_{\infty,q}$ in place of $\dot{B}^{s-\frac{d}{q}}_{\infty,\infty}$ when we want to control the $L^{p,q}$ norm. A finer example is needed order to show that this cannot be avoided. This example should contain oscillations in a wide range of frequencies in order to make the distinction between $\dot{B}^{s-\frac{d}{q}}_{\infty,q}$ and $\dot{B}^{s-\frac{d}{q}}_{\infty,\infty}$, and should also contain a wide range of magnitude in order to distinguish between L^p and $L^{p,q}$. Our example has roughly the form of a hyperbolic "chirp"

$$f(x) = x^{-\alpha}\sin(\frac{1}{x}),$$

with $\alpha > 0$. For the sake of simplicity, we have placed ourselves in the one-dimensional case $d = 1$ although this example can be generalized to arbitrary dimensions. In order to simplify our analysis, we shall define a variant of f in the form of a wavelet series. We recall that the Haar system is defined by the functions

$$\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k), \quad j,k \in \mathbb{Z},$$

where $\psi := \chi_{[0,\frac{1}{2}[} - \chi_{[\frac{1}{2},1]}$. Besov spaces $\dot{B}^s_{p,q}$ can be characterized as follows by wavelets (see e.g. [14] or [6]): if $f = \sum_{j,k} d_{j,k}\psi_{j,k}$, then

$$\|f\|_{\dot{B}^s_{p,q}} \sim \|\left(2^{(s+\frac{1}{2}-\frac{1}{p})j}\|(d_{j,k})_{k\in\mathbb{Z}}\|_{\ell^p}\right)_{j\in\mathbb{Z}}\|_{\ell^q}.$$

In the case of the Haar system, this characterisation holds if $\frac{1}{p}-1 < s < \frac{1}{p}$ (due to the lack of smoothness of the functions $\psi_{j,k}$) which is sufficient for our purpose.

Let $q \geq 1$ and $0 < s < \frac{1}{q}$, and let $\frac{1}{p} := \frac{1}{q} - s$. For $J > 0$, we consider the function

$$f_J(x) = \sum_{j=1}^{J} 2^{(\frac{1}{p}-\frac{1}{2})j}\psi_{j,1}(x) = \sum_{j=1}^{J} 2^{\frac{j}{p}}\psi(2^j x - 1). \tag{4.11}$$

Note that the wavelets in the expansion of f_J have disjoint supports. We first notice that we have

$$\|f_J\|_{\dot{B}^s_{q,q}} \sim \left(\sum_{j=1}^{J} 2^{(qs+\frac{q}{2}-1)j}2^{(\frac{q}{p}-\frac{q}{2})j}\right)^{\frac{1}{q}} = J^{\frac{1}{q}}.$$

For the Besov norms of negative indices, we have

$$\|f_J\|_{\dot{B}^{s-\frac{1}{q}}_{\infty,\infty}} \sim \sup_{j=1}^{J} 2^{(s-\frac{1}{q}+\frac{1}{2})j}2^{(\frac{1}{p}-\frac{1}{2})j} = 1,$$

and

$$\|f_J\|_{\dot{B}^{s-\frac{1}{q}}_{\infty,q}} \sim \left(\sum_{j=1}^{J}[2^{(s-\frac{1}{q}+\frac{1}{2})j}2^{(\frac{1}{p}-\frac{1}{2})j}]^q\right)^{\frac{1}{q}} = J^{\frac{1}{q}}.$$

Since the wavelets in the expansion of f_J have disjoint supports, the L^p norm is of the order

$$\|f_J\|_{L^p} = \left(\sum_{j=1}^{J} \|2^{\frac{j}{p}}\psi(2^j\cdot -1)\|^p_{L^p}\right)^{\frac{1}{p}} \sim J^{1/p}.$$

Remarking that the rearrangement function of f_J is simply given by

$$f_J^*(s) = |f_J(s + 2^{-J})|,$$

we can also easily evaluate its $L^{p,q}$ norm using (2.5) which gives

$$\begin{aligned}\|f_J\|_{L^{p,q}} &= \left(\sum_{j=1}^{J} 2^{\frac{jq}{p}} \int_0^{2^{-j}} (s + 2^{-j} - 2^{-J})^{\frac{q}{p}-1} ds\right)^{\frac{1}{q}} \\ &\sim \left(\sum_{j=1}^{J} 2^{\frac{jq}{p}} 2^{-j} 2^{-j(\frac{q}{p}-1)}\right)^{\frac{1}{q}} \\ &= J^{\frac{1}{q}}.\end{aligned}$$

From all these estimates, we obtain that

$$\|f_J\|^{1-qs}_{\dot{B}^s_{q,q}}\|f_J\|^{qs}_{\dot{B}^{s-\frac{1}{q}}_{\infty,\infty}} \sim J^{\frac{1}{q}-s} = J^{\frac{1}{p}} \sim \|f_J\|_{L^p},$$

and

$$\|f_J\|^{1-qs}_{\dot{B}^s_{q,q}}\|f_J\|^{qs}_{\dot{B}^{s-\frac{1}{q}}_{\infty,q}} \sim J^{\frac{1}{q}-s}J^s = J^{\frac{1}{q}} \sim \|f_J\|_{L^{p,q}}.$$

Again (1.3) and (1.1) are sharp, but this example also shows that the use of $\dot{B}^{s-\frac{d}{q}}_{\infty,q}$ in place of $\dot{B}^{s-\frac{d}{q}}_{\infty,\infty}$ is unavoidable for the validity of (1.3).

5 Refined Hardy inequalities

Let as in the previous sections $1 < p < \infty$ and $s \in]0, \frac{d}{q}[$, with $\frac{1}{p} = \frac{1}{q} - \frac{s}{d}$. For such indices, we claim that the refined Hardy inequality (1.4) holds. It is actually a direct consequence of (1.3) combined with a generalized version of Hölder's inequality known as O'Neil inequalities : if $1 \leq p_1, p_2, q_1, q_2 \leq \infty$, then for any $f \in L^{p_1,q_1}(\mathbb{R}^d)$ and $g \in L^{p_2,q_2}(\mathbb{R}^d)$,

$$\|fg\|_{L^{p,q}(\mathbb{R}^d)} \leq C\|f\|_{L^{p_1,q_1}(\mathbb{R}^d)}\|g\|_{L^{p_2,q_2}(\mathbb{R}^d)}, \tag{5.12}$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ and $C = C(p_1, p_2, q_1, q_2)$ (a first proof of these inequalities using rearrangements can be found in [16], while a more modern proof would proceed by bilinear interpolation between the usual inequalities for Lebesgue spaces).

We now take $g(x) = \frac{1}{|x|^s}$ and apply (5.12), in the specific form

$$\|fg\|_{L^{q,q}(\mathbb{R}^d)} \leq C\|f\|_{L^{p,q}(\mathbb{R}^d)}\|g\|_{L^{r,\infty}(\mathbb{R}^d)}$$

where $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$, i.e. $r = \frac{d}{s}$. Since obviously $g \in L^{r,\infty}(\mathbb{R}^d)$, we have

$$\Big(\int \frac{|f(x)|^q}{|x|^{sq}}\Big)^{\frac{1}{q}} \leq C\|f\|_{L^{p,q}(\mathbb{R}^d)}.$$

Combining this with (1.3), we obtain (1.4). Let us mention that this inequality is obtained in [1] for $q = 2$ by a different method which can not be generalized to the general L^q framework.

Concerning the sharpness of (1.4), we can make similar observations as for (1.3). In particular, this inequality is sharp for oscillatory functions of the type $\varphi_\omega(x) = e^{i\omega x_1}\varphi(x)$. It is also easily checked that the sequence f_J defined by (4.11) satisfies

$$\Big(\int \frac{|f_J(x)|^q}{|x|^{sq}}\Big)^{\frac{1}{q}} \sim J^{\frac{1}{q}} \sim \|f_J\|_{L^{p,q}(\mathbb{R}^d)},$$

making (1.4) sharp for such functions, and justifying the use of $\dot{B}^{s-\frac{d}{q}}_{\infty,q}$ in place of $\dot{B}^{s-\frac{d}{q}}_{\infty,\infty}$.

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