Universidad de Chile — The numerical simulation of complex PDE problems
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Homework N° 1

The finite element method for a mixed problem

The Stokes problem in $\mathbb{R}^d$

We consider an open bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) and we denote $\partial\Omega$ its boundary, supposed sufficiently smooth. We first introduce the functional spaces :

$$L_0^2(\Omega) = \{q \in L^2(\Omega)\,,\ \int_\Omega q\,dx = 0\}\,,$$
$$H_{div}^1(\Omega)^d = \{v \in H_0^1(\Omega)^d\,,\ \operatorname{div} v = 0 \text{ in } \Omega\}\,.$$

and we provide the following result as a corollary of DeRham theorem :

Corollary 1 *Suppose $f \in H^{-1}(\Omega)^d$ is such that $\langle f, v\rangle = 0$ for all $v \in H_{div}^1(\Omega)^d$. Then, there exists $p \in L^2(\Omega)$ such that $f = \nabla p$. Furthermore, if Ω is a connected set, then p is uniquely defined up to an additive constant and we have the following estimate :*

$$\|p\|_{L^2(\Omega)} \le C\|f\|_{H^{-1}(\Omega)}\,.$$

Part I : *Variational formulation*

We investigate the Stokes problem describing the behavior of an incompressible viscous flow confined in Ω :
Given $f \in L^2(\Omega)^d$ and $\nu \in \mathbb{R}_+^*$, find $u \in H^1(\Omega)^d$ and $p \in L^2(\Omega)$ such that :

$$(\mathcal{S}) \begin{cases} -\nu\Delta u + \nabla p = f\,, & \text{in } \Omega \\ \operatorname{div} u = 0\,, & \text{in } \Omega \\ u = 0\,, & \text{on } \partial\Omega \end{cases} \qquad (1)$$

The unknowns are the velocity components in u and the pressure p. Notice that p is only known via its gradient, so it will be defined up to a constant, the latter being fixed by the condition $p \in L_0^2(\Omega)$.

1. Show that the problem (1) is equivalent to the following problem :
Find $(u, p) \in (H_0^1(\Omega)^d, L_0^2(\Omega))$ such that :

$$\begin{cases} \nu \displaystyle\int_\Omega \nabla u : \nabla v\,dx - \int_\Omega p \operatorname{div} v\,dx = \int_\Omega f \cdot v\,dx\,, & \forall v \in H_0^1(\Omega)^d \\ \operatorname{div} u = 0 & \text{in } \Omega. \end{cases} \qquad (2)$$

2. We introduce the following formulation :
   Find $u \in H^1_{div}(\Omega)^d$ *such that :*

$$\nu \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f \cdot v \, dx \,, \qquad \forall v \in H^1_{div}(\Omega)^d \,. \tag{3}$$

   (a) Show that there exists a unique solution u to the problem (3), that depends continuously on the right-hand side term f.
   (b) Deduce from Corollary 1 that there exists a unique $p \in L^2_0(\Omega)$ such that (u, p) is the unique solution of the problem (2), where u is the solution of problem (3). Show that (u, p) depends continuously on f.

Part II : *Numerical approximation*

We consider an abstract functional context for the Stokes problem.
Let X and M be two Hilbert spaces with their associated norms $\|\cdot\|_X$ and $\|\cdot\|_M$. We introduce two continuous bilinear forms, a defined on $X \times X$ and b defined on $X \times M$ such that the Stokes problem (1) reads :
Given $f \in X'$, *find* $(u, p) \in X \times M$ *such that :*

$$(\mathcal{S}) \begin{cases} a(u, v) + b(v, p) = f(v) \,, & \forall v \in X \\ b(u, q) = 0 \,, & \forall q \in M \,. \end{cases} \tag{4}$$

1. Identify the two bilinear forms a and b corresponding to the Stokes problem (1).
2. Show that the linear mapping $Au : v \mapsto a(u, v)$ is an element of X' and that $A \in \mathcal{L}(X, X')$. Show that the linear mapping $Bu : q \mapsto b(u, q)$ is an element of M' and that $B \in \mathcal{L}(X, M')$.
3. We introduce the space :

$$V = \{v \in X \,,\ b(v, q) = 0 \,,\ \forall q \in M\} = Ker(B) \,,$$

   and we consider the problem :
   Find $u \in V$ *such that :*

$$(\mathcal{P}) \qquad a(u, v) = f(v) \,, \qquad \text{for all } v \in V \,. \tag{5}$$

   Under what condition(s) the problem (4) above has a unique solution $(u, p) \in X \times M$ such that u is solution of problem (5) ?

Approximation of ($\mathcal{S}$).
Let h denotes a discretization parameter. Let $X_h \subset X$ and $M_h \subset M$ be two finite dimensional subspaces. We introduce the following approximation problems :
Find $(u_h, p_h) \in X_h \times M_h$ *such that*

$$(\mathcal{S}_h) \begin{cases} a(u_h, v_h) + b(v_h, p_h) = f(v_h) \,, & \forall v_h \in X_h \\ b(u_h, q_h) = 0 \,, & \forall q_h \in M_h \,. \end{cases} \tag{6}$$

Find $u_h \in V_h$ such that :

$$(\mathcal{P}_h) \qquad a(u_h, v_h) = f(v_h)\,, \qquad \forall v_h \in V_h \tag{7}$$

where

$$V_h = \{v_h \in X_h\,,\ b(v_h, q_h) = 0\,,\ \forall q_h \in M_h\}\,.$$

Notice that there is generally no relation between V and V_h ($V_h \not\subset V$).

4. Show that if a is V-elliptic and if the following discrete *inf-sup* condition (LBB condition)

$$\exists \beta_h > 0\,, \qquad \inf_{q_h \in M_h} \sup_{v_h \in X_h} \frac{b(v_h, q_h)}{\|v_h\|_X \|q_h\|_M} \geq \beta_h\,,$$

   is satisfied then the problem $(\mathcal{S}_h)$ has a unique solution $(u_h, p_h) \in X_h \times M_h$ where u_h is the solution of the problem $(\mathcal{P}_h)$.

5. Here, we assume the two formulations are equivalent. Show that for all $v_h \in V_h, q_h \in M_h$:

$$a(u_h - v_h, u_h - v_h) = a(u - v_h, u_h - v_h) - \langle p - q_h, \mathrm{div}(u_h - v_h)\rangle_{L^2(\Omega)}$$

6. Deduce an estimate of $\|u_h - v_h\|_X$ and that we have :

$$\|u - u_h\|_X \leq C \left(\inf_{v_h \in V_h} \|u - v_h\|_X + \inf_{q_h \in M_h} \|p - q_h\|_M \right)\,.$$

   On what parameter depends the constant C ?

7. Show that if the *inf-sup* condition above is satisfied, then :

$$\|p_h - q_h\|_M \leq C_1 \|q_h - p\|_M + C_2 \|u - u_h\|_X\,, \qquad \forall q_h \in M_h\,.$$

8. Deduce the estimate :

$$\|u - u_h\|_X + \|p - p_h\|_M \leq C_1 \inf_{v_h \in V_h} \|u - v_h\|_X + C_2 \inf_{q_h \in M_h} \|p - q_h\|_M\,.$$

   Compute the constants C_1 and C_2.

Part III : *Finite element approximation*

We consider the Stokes problem (1) posed in $\mathbb{R}^2$. The domain $\Omega =]0,1[^2$ is supposed to be covered by a mesh T_h, where the parameter h denotes the grain of the discretization : $h = \max_{K \in T_h} diam(K)$. We introduce the following approximation spaces :

$$X_h = \{v_h \in C^0(\bar{\Omega})^2\,,\ v_h|_K \in \mathbb{P}_1\,,\ \forall K \in T_h \text{ and } v_h|_{\partial\Omega} = 0\}$$
$$M_h = \{q_h \in L^2(\Omega)\,,\ q_h|_K \in \mathbb{P}_0\,,\ \forall K \in T_h \text{ and } \int_\Omega q_h dx = 0\}\,.$$

9. Show that $X_h \subset X = H_0^1(\Omega)^2$, and $M_h \subset M = L_0^2(\Omega)$.

10. Show that these approximation spaces do not satisfy the discrete *inf-sup* condition for the bilinear form b of the Stokes problem. To this end, show that there exists $p_h \neq 0 \in M_h$ such that $\int_\Omega p_h div v_h \, dx = 0$, for all $v_h \in X_h$.
11. To circumvent this problem, we introduce another approximation space, between $\mathbb{P}_1^2$ and $\mathbb{P}_2^2$. We consider the polynomial space :

$$P(K) = (\mathbb{P}_1 \oplus \mathrm{Span}\{p_1, p_2, p_3\})^2$$

where

$$p_1 = n_1 \lambda_2 \lambda_3 \,, \qquad p_2 = n_2 \lambda_3 \lambda_1 \,, \qquad p_3 = n_3 \lambda_1 \lambda_2 \,,$$

and n_i denotes the outer normal to edge e_i in triangle K and the λ_i are the barycentric coordinates of the mid-points of the edges. The degrees of freedom associated with $P(K)$ are defined as :

$$\Sigma = \{v \to v(a_i) \,, \;\; 1 \leq i \leq 3 \,, \; v \to \int_{e_i} v \cdot n_i \, dx \,, \;\; 1 \leq i \leq 3\} \,,$$

where $(a_i)_{1 \leq i \leq 3}$ denote the vertices of K.

(a) Check that the $(p_i)_{1 \leq i \leq 3}$ are linearly independent and that $P(K)$ is a direct sum.

(b) Show that $(K, P(K), \Sigma)$ is a finite element and determine the shape functions associated with the vertices a_i of K and with the other degrees of freedom.

(c) Define the interpolation operator Π_K for the finite element $(K, P(K), \Sigma)$. Show that

$$v \in P(K) \;\Rightarrow\; \Pi_K v = v \,.$$

(d) We keep the space M_h and we consider now a new approximation space :

$$X_h = \{v_h \in C^0(\bar{\Omega})^2 \,, \; v_h|_K \in P(K) \,, \; \forall K \in T_h \text{ and } v_h|_{\partial\Omega} = 0\} \subset X = H_0^1(\Omega)^2 \,.$$

Suppose the discrete *inf-sup* condition is satisfied for (M_h, X_h) with respect to the bilinear form b of the Stokes problem. Provide an estimate of :

$$\inf_{w_h \in X_h} |v - w_h|_{H^1(\Omega)} \,, \qquad \text{where} \quad |u|_{H^1(\Omega)} = \left(\int_\Omega (\nabla u)^2 \right)^{1/2}$$

for $v \in X \cap H^2(\Omega)^2$, when the mesh T_h is uniform of size h.

(e) We consider the orthogonal projection operator P_K from $L^2(K)$ onto $\mathbb{P}_0$:

$$\int_K (P_K q - q) \, dx = 0 \,, \qquad \text{with } P_K q \in \mathbb{P}_0 \quad \text{and } q \in L^2(K) \,.$$

Prove that if $q \in H^1(K)$ then

$$\|q - P_K q\|_{L^2(K)} \;\leq\; Ch |q|_{H^1(K)}$$

where C is independent of K. Let $P_h : M \to M_h$ be the operator such that :

$$P_h q|_K = P_K q \,.$$

Verify that $P_h q \in M_h$ if $q \in M$. Deduce from the previous question that :

$$\inf_{q_h \in M_h} \|p - q_h\|_{L^2(\Omega)} \;\leq\; Ch |p|_{H^1(\Omega)} \,, \qquad \text{if } p \in H^1(\Omega) \,.$$

(f) What is the purpose of having estimates on $\inf_{w_h \in X_h} |v - w_h|_{H^1(\Omega)}$ and $\inf_{q_h \in M_h} \|p - q_h\|_{L^2(\Omega)}$?