Chapter 1

Distributions on open sets of $\mathbb{R}^d$

"...For the Fourier transform, the introduction of distribution (hence the space $S$) is inevitable either in an explicit or hidden form...As a result one may obtain all that is desired from the point of view of the continuity and inversion of the Fourier transform."
Laurent Schwartz (1915-2002)

The theory of distributions has been introduced by L. Schwartz (1915-2002) ca. 1944-1950 as "objects" which generalize functions [Schwartz, 1945, Schwartz, 1966]. They extend the notion of derivative to all integrable functions and are now widely used to formulate generalized solutions of partial differential equations. The Lebesgue $L^p$ spaces contain non regular and non continuous functions for which derivatives are not defined in the classical sense. Nonetheless, the classical derivatives generally exist almost everywhere. It seems then reasonable to generalize the notion of derivative to be independent of zero-measure subsets. This leads to the concept of weak derivative introduced by J. Leray (1906-1998) and S.L. Sobolev (1908-1989). The main concept underlying the notion of weak derivative is the concept of distributions.

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According to [Schwartz, 1966], O. Heaviside (1850-1925) introduced in 1894 a function denoted $H$ and called the unit step function. It corresponds to the characteristic function of the interval $[0, +\infty]$ of $\mathbb{R}$ and is a discontinuous function whose value is zero for negative argument and one for positive argument (cf. Figure 1.1). This function has no derivative at the origin. Nonetheless, Heaviside defined the derivative $H'$ of $H$ at any point in $\mathbb{R}$, and called it the unit impulse function, as:

$$H'(x) = \begin{cases} 
0 & x \neq 0 \\
+\infty & x = 0 
\end{cases} \quad (1.1)$$

This function $H$ cannot be considered as a usual function. Since $H'$ vanishes almost everywhere, it should be assigned integral zero:

$$\int_{-\alpha}^{\alpha} H'(x) \, dx = 0, \quad \forall \alpha \in ]0, +\infty[. \quad (1.2)$$
Furthermore, we have:
\[
\int_{-a}^{a} H'(x) \, dx = H(a) - H(-a) = 1, \quad \forall a \in \mathbb{R}.
\]  
(1.3)

Actually, the function \( H' \) has been reintroduced by P. Dirac (1902-1984) in 1926 and is widely used in the context of quantum physics, where it is called the Dirac delta function, denoted \( \delta \). It has the following property:
\[
\int_{-\infty}^{+\infty} f(x) \delta(x) \, dx = f(0),
\]
for all continuous functions \( f \). However, there is no function verifying this property.

The purpose of the distribution theory is to eradicate all these contradictions by giving a rigorous definition of generalized functions, called distributions. Distributions have the remarkable property of being all indefinitely differentiable (in a certain sense). Furthermore, an noteworthy property is that the derivation is a continuous operation in the distribution space.

1.1 Infinitely smooth functions

In this section, we review some basic notions of differential calculus and we introduce the definition of function with compact support\(^1\).

1.1.1 Differential calculus

Let consider an open subset \( \Omega \subset \mathbb{R}^d \). We denote by \( \mathcal{L}(\mathbb{R}^d, \mathbb{R}^m) \) the vector space of linear maps. This space becomes a Banach space when endowed with the norm:
\[
\|u\|_{\mathcal{L}(\mathbb{R}^d, \mathbb{R}^m)} = \sup_{\|x\|<1} \|ux\|, \quad \forall u \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^m).
\]

\(^1\)This presentation has been inspired by various lecture notes, in particular those of J.-M. Bony, Analyse, Ecole Polytechnique, Palaiseau, of M. Tucsnak, Distributions et équations fondamentales de la physique, Institut Elie Cartan, Université Nancy I and of J.-Y. Chemin, Analyse réelle, UPMC, Paris.
1.1. Infinitely smooth functions

**Definition 1.1** The function \( f : \Omega \subset \mathbb{R}^d \to \mathbb{R}^m \) is said to be differentiable at \( x \in \Omega \) if there exists an element \( f'(x) \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^m) \) such that
\[
\| f(x + h) - f(x) - f'(x)h \| = o(\|h\|), \quad h \to 0.
\]

**Lemma 1.1** If \( f : \Omega \to \mathbb{R} \) is differentiable at \( x_0 \), then \( f \) has a partial derivative with respect to \( x_j \) at \( x_0 \) and
\[
\frac{\partial f}{\partial x_j}(x_0) = f'(x_0)e_j, \quad \forall j \in \{1, \ldots, d\},
\]
where \((e_1, \ldots, e_d)\) denotes the canonical basis of \( \mathbb{R}^d \). Moreover, we have
\[
f'(x_0)v = \langle \nabla f(x_0), v \rangle, \quad \forall v \in \mathbb{R}^d,
\]
the vector \( \nabla f(x_0) = \sum_{j=1}^d \frac{\partial f}{\partial x_j}(x_0)e_j \) is the gradient of \( f \) at \( x_0 \).

**Proposition 1.1** Suppose \( I \) is an open interval of \( \mathbb{R} \) and \( f \) is continuous in \( I \) and differentiable except at point \( x_0 \in I \). If \( x \in I \) and \( \lim_{x \to x_0} f'(x) = a \in \mathbb{R} \), then \( f'(x_0) \) exists and we have \( f'(x_0) = a \).

**Definition 1.2** Consider an open subset \( \Omega \subset \mathbb{R}^d \) and \( k \in \mathbb{N} \). A function \( f \in C^k(\Omega) \) if one of the following conditions holds:

(i) \( k = 0 \) and \( f \) is continuous on \( \Omega \),

(ii) \( k \geq 1 \) and \( f \) has partial derivatives \( \frac{\partial f}{\partial x_j} \), \( j = 1, \ldots, d \) that are in \( C^{k-1}(\Omega) \).

Moreover, \( f \in C^\infty(\Omega) \) if \( f \in C^k(\Omega) \) for all \( k \in \mathbb{N} \).

**Lemma 1.2 (Schwarz)** If \( f \in C^2(\Omega) \) then
\[
\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}, \quad \forall i, j \in \{1, \ldots, d\}.
\]

Notice that if \( f \in C^k(\Omega) \), this lemma refers to the property of interchanging the order of taking partial derivatives of a function. The matrix of second-order partial derivatives of \( f \) is called the Hessian matrix. In most cases, this matrix is a symmetric matrix.

**Theorem 1.1 (Taylor’s formula)** Let \( \Omega \) be an open subset of \( \mathbb{R}^d \), \( m \in \mathbb{N} \) and \( f \in C^{m+1}(\Omega) \). Consider \( x, y \in \Omega \) such that \( x + ty \in \Omega \), for all \( t \in [0,1] \). Then, we have:
\[
f(x + y) = \sum_{|\alpha| \leq m} \frac{y^\alpha}{\alpha!} \partial^\alpha f(x) + (m + 1) \sum_{|\gamma|=m+1} \frac{y^\gamma}{\gamma!} \int_0^1 (1-t)^m \partial^\gamma f(x + ty) \, dt.
\]

**Corollary 1.1** Let \( \Omega \) be an open subset of \( \mathbb{R}^d \), \( f \in C^{m+1}(\Omega) \) and \( K \) a compact subset of \( \Omega \). Consider \( x, y \in K \) such that \( [x, x + y] \subset K \). Then, there exists a constant \( C = C(K, m, f) \) such that
\[
\left| f(x + y) - \sum |\alpha| \leq m \frac{y^\alpha}{\alpha!} \partial^\alpha f(x) \right| \leq C\|y\|^{m+1}.
\]

**Corollary 1.2** If \( f \in C^k(B) \) where \( B = \{x \in \mathbb{R}^d, \|x\| < 1\} \), then there exists \( f_1, \ldots, f_d \in C^{k-1}(B) \) such that:
\[
\partial^\alpha f_j(0) = \frac{\partial^\alpha \partial_j f(0)}{1 + |\alpha|}, \quad \sup_{x \in B} |\partial^\alpha f_j| \leq \sup_{x \in B} |\partial^\alpha \partial_j f|, \forall j = 1, \ldots, d \quad f(x) - f(0) = \sum_{j=1}^d x_j f_j(x).
\]
1.1.2 Existence of $C^\infty$ functions

Let $\Omega \subset \mathbb{R}^d$ be an open set. We recall that the set of infinitely smooth functions is defined as:

$$C^\infty(\Omega) = \bigcap_{k \geq 0} C^k(\Omega).$$

**Definition 1.3** Given $u \in C^0(\Omega)$, the support of $u$, denoted as $\text{Supp}(u)$, is defined as the closure of the set $\{ x \in \Omega, u(x) \neq 0 \}$ in $\Omega$. It is the smallest closed subset of $\Omega$ such that $u = 0$ in $\Omega \setminus \text{Supp}(u)$.

Notice that if $x \in \text{Supp}(u)$, then there exists a sequence $(x_n)_{n \in \mathbb{N}} \subset \Omega$ such that $u(x_n) \neq 0$ for all $n \in \mathbb{N}$ and such that $\lim_{n \to \infty} x_n = x$.

**Definition 1.4** If $k \in \mathbb{N} \cup \{+\infty\}$, the space $C^k_0(\Omega)$ is composed of all functions $u \in C^k(\Omega)$ having a compact subset of $\Omega$ as support. The elements of $C^\infty_c(\Omega)$, hereafter denoted by $\mathcal{D}(\Omega)$, are called test functions.

It is common to find the notations $C^{\infty}_c(\Omega)$ or $C^\infty(\Omega)$ instead of the symbol $\mathcal{D}(\Omega)$. Every function $u \in C^k_0(\Omega)$ can be extended to a function of $C^k_0(\mathbb{R}^d)$. Thus, $C^k_0(\Omega)$ can be seen as a subspace of $C^k_0(\mathbb{R}^d)$. In this respect, given an open set $\Omega \subset \mathbb{R}^d$, the set $C^k_0(\Omega)$ can be defined as the set of elements $u \in C^k_0(\mathbb{R}^d)$ for which $\text{Supp}(u) \subset \Omega$.

The space $\mathcal{D}(\Omega)$ is not empty. Indeed, we have the following result.

**Lemma 1.3** There exists a function $u \in \mathcal{D}(\mathbb{R}^d)$ such that $u(0) > 0$ and $u(x) \geq 0$, for all $x \in \mathbb{R}^d$.

**Proof.** Consider the function $f \in C^\infty(\mathbb{R})$ defined as:

$$f(x) = \begin{cases} 0 & x \leq 0 \\ \exp(-1/x) & x > 0 \end{cases}$$

Then, the function

$$u(x) = f(1 - \|x\|^2)$$

satisfies the assumptions. By translation and scaling, we show that for every $r > 0$, the function

$$x \mapsto u\left(\frac{x-x_0}{r}\right)$$

is positive on $\mathbb{R}$, strictly positive at $x_0$ and its support is the ball of radius $r$ centered at $x_0$. $\square$

The existence of such function $u$ allows to prove a classical result.

**Theorem 1.2** If $f, g \in C^0(\Omega)$ and if the following identity holds

$$\int_{\Omega} f u \, dx = \int_{\Omega} g u \, dx, \quad \forall u \in \mathcal{D}(\Omega),$$

then, $f = g$.

**Proof.** Let consider $h = f - g$, then

$$\int_{\Omega} h u \, dx = 0, \quad \forall u \in \mathcal{D}(\Omega).$$

If $h$ is a complex valued function, we will consider the real and imaginary parts separately. Hence, consider $h$ is a real valued function and that the previous equality holds for all $u \in \mathcal{D}(\Omega)$, $u$ being a real valued function. If there exists $x_0$ such that $h(x_0) \neq 0$ then we select $u$ such that $u(x_0) > 0$ with support in a neighborhood of $x_0$ and such that $uh$ has constant sign. This obviously is in contradiction with the assumption, thus $h \equiv 0$ in $\Omega$. $\square$

**Lemma 1.4** There exists an increasing function $\theta \in C^\infty(\mathbb{R})$ such that

$$\theta(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x \geq 1 \end{cases}$$
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Proof. Consider the function \( f \in C^\infty(\mathbb{R}) \) defined hereabove as:

\[
f(x) = \begin{cases} 
0 & x \leq 0 \\
\exp(-1/x) & x > 0 
\end{cases}
\]

In this case, \( \text{Supp}(f) = [0, \infty[ \) and \( 0 \leq f(x) \leq 1 \), for all \( x \in \mathbb{R} \). We introduce the functions \( g(x) = f(x)f(1-x) \) and \( G(x) = \int_0^x g(t) \, dt \). We notice that \( 0 \leq g(x) \leq 1 \), for all \( x \in \mathbb{R} \) and that \( \text{Supp}(g) \subset [0, 1] \) and \( g \neq 0 \). The function \( \theta = \frac{G(x)}{G(1)} \in C^\infty(\mathbb{R}) \) is an increasing function and is such that

\[
\theta(x) = \begin{cases} 
0 & x \leq 0 \\
1 & x \geq 1 
\end{cases}
\]

\( \square \)

Proposition 1.2 Consider four real numbers \( a, b, c \) and \( d \) such that \( a < c < d < b \). Then, there exists \( \rho \in \mathcal{D}(\mathbb{R}) \) such that:

(i) \( \rho(x) = 1 \), for all \( x \in ]c, d[ \),

(ii) \( \text{Supp}(\rho) \subset ]a, b[ \),

(iii) \( 0 \leq \rho(x) \leq 1 \), for all \( x \in \mathbb{R} \).

Proof. Consider the function \( \rho(x) = \theta \left( \frac{x-a}{c-a} \right) \theta \left( \frac{b-x}{d-b} \right) \), where \( \theta \) is the function defined in the previous lemma.

If \( K \) is a compact subset of \( \mathbb{R}^d \) and \( \varepsilon > 0 \), we denote

\[
K_\varepsilon = K + B(0, \varepsilon) = \bigcup_{x \in K} B(x, \varepsilon),
\]

where \( B(x, \varepsilon) \) the ball of radius \( \varepsilon \) centered at \( x \in \mathbb{R}^d \).

Proposition 1.3 Consider a compact set \( K \subset \mathbb{R}^d \). Then, for every \( \varepsilon > 0 \), there exists \( u \in \mathcal{D}(K_{2\varepsilon}) \) such that:

\[
\begin{cases} 
u(x) = 1 & \forall x \in K_\varepsilon \\
0 \leq \nu(x) \leq 1 & \forall x \in \mathbb{R}^d 
\end{cases}
\]

Proof. By compacity, there exists \( x_1, \ldots, x_p \in K \) such that

\[
K_\varepsilon \subset \bigcup_{j=1}^p B \left( x_j, \frac{4\varepsilon}{3} \right).
\]

For each \( j = 1, \ldots, p \) there exists a function \( u_j \in \mathcal{D}(B(x_j, \frac{\varepsilon}{3})) \) such that \( u_j(x) \geq 0 \) for all \( x \in \mathbb{R}^d \) and \( u_j \equiv 1 \) on \( B(x_j, \frac{\varepsilon}{3}) \). Consider \( \tilde{u}(x) = \sum_{j=1}^p u_j(x) \). Then, \( \tilde{u}(x) \geq 1 \) for all \( x \in \bigcup_{j=1}^p B(x_j, \varepsilon) \). Moreover, given that

\[
\bigcup_{j=1}^p B \left( x_j, \frac{5\varepsilon}{3} \right) \subset K_{2\varepsilon},
\]

we conclude that \( \tilde{u} \in \mathcal{D}(K_{2\varepsilon}) \). Taking \( u = \theta(\tilde{u}(x)) \) where \( \theta \in C^\infty(\mathbb{R}) \) is the function introduced in the previous lemma, yields the result.

\( \square \)
Corollary 1.3 Let $K_1, K_2$ be two disjoined compact subsets of the open set $\Omega \subset \mathbb{R}^d$. Then, there exists a function $u \in \mathcal{D}(\Omega)$ such that
\[
u(x) = \begin{cases} 
1 & \forall x \in K_1 \\
-1 & \forall x \in K_2
\end{cases}
\]
and such that $|u(x)| \leq 1$ for all $x \in \Omega$.

Proof. Consider $U_1$ and $U_2$ two open sets of $\Omega$ such that
\[K_1 \subset U_1, \quad K_2 \subset U_2, \quad U_1 \cap U_2 = \emptyset.\]
From the previous proposition, we know that there exists $u_1, u_2 \in \mathcal{D}(\Omega)$ such that:
\[u_i \equiv 1, \text{ in } K_i, \quad u_i \in \mathcal{D}(U_i), \quad 0 \leq u_i(x) \leq 1, \quad i \in \{1, 2\}.
\]
The function $u$ defined as: $u(x) = u_1(x) - u_2(x)$, $\forall x \in \Omega$ satisfies the desired properties. \qed

1.1.3 Partition of unity

Partitions of unity are useful because they often allow one to extend local constructions to the whole space.

Proposition 1.4 Let $K$ be a compact subset of $\mathbb{R}^d$ and let consider an open cover $(U_j)_{j=1,\ldots,N}$ of $K$. Then, there exists compact sets $(K_j)_{j=1,\ldots,N}$ such that $K_j \subset U_j$, for all $j = 1, \ldots, N$ and
\[K = \bigcup_{j=1}^{N} K_j.\] (1.6)

Proof. For every $x \in K$, consider $r_x > 0$ such that $\overline{B(x, r_x)} \subset \bigcap_{x \in U_j} U_j$. Hence, we have $K \subset \bigcup_{x \in K} B(x, r_x)$ and thus there exists $x_1, \ldots, x_M \in K$ such that $K \subset \bigcup_{i=1}^{M} B(x_i, r_{x_i})$. We pose
\[K_j = K \cap \left( \bigcup_{B(x_i, r_{x_i}) \subset U_j} B(x_i, r_{x_i}) \right).
\]
Then by definition, $K_j$ is a compact set included in $K$ such that $K_j \subset U_j$. Let consider $x \in K$. There exists $i \in \{1, \ldots, M\}$ such that $x \in B(x_i, r_{x_i})$. Moreover, there exists $j_0 \in \{1, \ldots, N\}$ such that $x_i \in U_{j_0}$, thus $\overline{B(x_i, r_{x_i})} \subset U_{j_0}$. We conclude that $x \in K_{j_0}$. \qed

Theorem 1.3 (Partition of unity) Consider a compact subset $K$ of $\mathbb{R}^d$ and open sets $U_j$ of $\mathbb{R}^d$. Suppose we have $K \subset \bigcup_{j=1}^{N} U_j$, then there exists a collection $(u_j)_{j=1,\ldots,N}$ such that $u_j \in \mathcal{D}(U_j)$, $0 \leq u_j \leq 1$ for all $j = 1, \ldots, N$ and such that $\sum_{j=1}^{N} u_j = 1$ in the neighborhood of $K$. 

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Proof. We know that there exists compact sets \((K_j)_{j \in \{1, \ldots, N\}}\) such that \(K_j \subset U_j\), for every \(j = 1, \ldots, N\) and such that \(K = \bigcup_{j=1}^{N} K_j\). Moreover, from the previous proposition, we deduce that for every \(j = 1, \ldots, N\), there exists \(v_j \in \mathcal{D}(U_j)\) such that \(v_j(x) \in [0, 1]\), for all \(x \in \mathbb{R}^d\) and \(v_j(x) = 1\) if \(x \in K_j\). Consider the open set

\[
V = \left\{ x \in \bigcup_{j=1}^{N} U_j, \quad \sum_{j=1}^{N} v_j(x) > 0 \right\}.
\]

We have then \(K \subset V\). Hence, there exists \(\eta \in \mathcal{D}(V)\) such that \(\eta(x) \in [0, 1]\) for \(x \in \mathbb{R}^d\) and \(\eta \equiv 1\) on an open set \(W\) such that \(K \subset W \subset V\). Considering the function

\[
u_i = \frac{v_j}{(1 - \eta) + \sum_{k=1}^{N} v_k},
\]

we notice that \(u_i \in \mathcal{D}(U_j)\) as the denominator is strictly positive on \(V\) and is equal to \(1\) outside \(V\). Since \(\eta \equiv 1\) on the set \(W\), the relation implies that \(\sum_{j=1}^{N} \nu_i \equiv 1\) on \(W\). \(\square\)

Definition 1.5 Under the hypothesis of Theorem 1.3, the collection \((u_j)_{j=1, \ldots, N}\) is called a partition of unity subordinate to the open cover \((U_j)_{j=1, \ldots, N}\) of \(K\).

1.1.4 Lebesgue integration

We briefly recall some important results about Lebesgue integration and \(L^p\) spaces and we refer the reader to Appendix D for more details and results.

If \(\Omega\) is an open subset of \(\mathbb{R}^d\) and \(p \geq 1\), we denote by \(L^p(\Omega)\) the space of \(p\)-power integrable functions with values in \(\mathbb{R}\). The space \(L^p(\Omega)\) endowed with the norm

\[
\|f\|_{L^p(\Omega)} = \left( \int_{\Omega} |f(x)|^p \, dx \right)^{\frac{1}{p}}, \quad \forall f \in L^p(\Omega),
\]

is a Banach space. If \(p \geq 1\), we denote by \(L^p_{\text{loc}}(\Omega)\) the space of locally integrable functions \(f\) such that \(f \in L^p(K)\), for every compact \(K \in \Omega\).

Lemma 1.5 If \(f \in L^p_{\text{loc}}(\Omega)\) then, there exists a largest open set \(V\) in \(\Omega\) such that \(f|_{V} = 0\) almost everywhere in \(V\).

Definition 1.6 Consider \(\Omega \subset \mathbb{R}^d\) and \(f \in L^1_{\text{loc}}(\Omega)\). We call essential support of \(f\) the closed set \(\Omega \setminus V\) where \(V\) is the largest open \(V \subset \Omega\) such that \(f|_{V} = 0\) almost everywhere in \(V\).

Theorem 1.4 (Lebesgue dominated convergence) Let \((f_n)_{n \in \mathbb{N}}\) be a sequence of functions of \(L^1(\Omega)\), with \(\Omega\) an open set of \(\mathbb{R}^d\). Suppose that

(i) \(f_n(x) \to f(x)\) almost everywhere on \(\Omega\),

(ii) there exists a function \(g \in L^1(\Omega)\) such that for every \(n\), \(|f_n(x)| \leq g(x)\) almost everywhere in \(\Omega\).

Then, \(f \in L^1(\Omega)\) and \(\lim_{n \to \infty} \|f_n - f\|_{L^1(\Omega)} = 0\).

Theorem 1.5 (Lebesgue) Let \(\Omega\) be an open set of \(\mathbb{R}^d\), \(f: \mathbb{R}^p \times \Omega \to \mathbb{R}\) and \(k \in \mathbb{N}\). Suppose that

(i) the function \(f_k: \Omega \to \mathbb{R}\) defined by \(f_k(\lambda) = f(x, \lambda)\) belongs to \(C^k(\Omega)\) for every fixed \(x \in \mathbb{R}^p\),
(ii) for all \( \alpha \in \mathbb{N}^n \) such that \(|\alpha| \leq k\) we have

\[
\sup_{\lambda \in \Omega} |\partial^\alpha f_x(\lambda)| \leq w_\alpha(x),
\]

where \( w_\alpha \in L^1(\mathbb{R}^p) \).

Then, the function \( g(\lambda) = \int_{\mathbb{R}^p} f(x, \lambda) \, dx \) is a function of \( C^k(\Omega) \) and for every \( \lambda \in \mathbb{R} \) we have:

\[
\partial^\alpha g(\lambda) = \int_{\mathbb{R}^p} \partial^\alpha f(x, \lambda) \, dx, \quad \forall \alpha \in \mathbb{N}^n \ |\alpha| \leq k.
\]

Consider \( \Omega_1 \subset \mathbb{R}^{n_1}, \ \Omega_2 \subset \mathbb{R}^{n_2} \) two open sets and a measurable function \( F : \Omega_1 \times \Omega_2 \rightarrow \mathbb{C} \). We have two classical results.

**Theorem 1.6 (Tonelli)** Suppose that \( \int_{\Omega_2} |F(x,y)| \, dy < \infty \) for almost every \( x \in \Omega_1 \) and that

\[
\int_{\Omega_1} dx \int_{\Omega_2} |F(x,y)| \, dy < \infty.
\]

Then \( F \in L^1(\Omega_1 \times \Omega_2) \).

**Theorem 1.7 (Fubini)** Suppose \( F \in L^1(\Omega_1 \times \Omega_2) \). Then, we have \( F(x,y) \in L^1(\Omega_2) \) for almost every \( x \in \Omega_1 \) and \( \int_{\Omega_2} F(x,y) \, dy \in L^1(\Omega_1) \). Similarly, we have \( F(x,y) \in L^1(\Omega_1) \) for almost every \( y \in \Omega_2 \) and

\[
\int_{\Omega_1} F(x,y) \, dx \in L^1(\Omega_2).
\]

Furthermore, we have:

\[
\int_{\Omega_1} dx \int_{\Omega_2} F(x,y) \, dy = \int_{\Omega_2} dy \int_{\Omega_1} F(x,y) \, dx = \int_{\Omega_1 \times \Omega_2} F(x,y) \, dx dy.
\]

To conclude this section, we give the following density result.

**Theorem 1.8** (i) If \( p \geq 1 \), then the space \( C^0_0(\Omega) \) is dense in \( L^p(\Omega) \).

(ii) If \( 1 \leq p < \infty \), the space \( \mathcal{D}(\Omega) \) is dense in \( L^p(\Omega) \).

### 1.2 The concept of distribution

#### 1.2.1 Main definitions

Let \( \Omega \) be an open subset of \( \mathbb{R}^d \) and let \( K \) denote a compact of \( \Omega \). In this section, we call test function on \( \Omega \) any function of \( \mathcal{D}(\Omega) \) and we denote by \( \mathcal{D}_K(\Omega) \) the set of test functions with support in \( K \). Furthermore, we like to introduce the following notations: given \((\alpha, \beta) \in \mathbb{N}^d \times \mathbb{N}^d, \ x \in \mathbb{R}^d, \) and \( u \in C^\infty(\Omega) \):

\[
|\alpha| \overset{\text{def}}{=} \sum_{j=1}^d \alpha_j, \quad \alpha! \overset{\text{def}}{=} \prod_{j=1}^d \alpha_j!, \quad x^\alpha \overset{\text{def}}{=} \prod_{j=1}^d x_j^{\alpha_j}, \quad \partial^\alpha u \overset{\text{def}}{=} \prod_{j=1}^d \partial_j^{\alpha_j} u.
\]
Definition 1.7 We call distribution on $\Omega$ any linear form $T$ on $\mathcal{D}(\Omega)$ such that the following continuity property holds: for every compact set $K \subset \Omega$, there exists an integer $N$ and a constant $C_K$ such that:

$$|\langle T, u \rangle| \leq C_K \sup_{|\alpha| \leq N} \|\partial^\alpha u\|_{L^\infty}, \quad \forall u \in \mathcal{D}_K(\Omega)$$

(1.8)

and we denote by $\mathcal{D}'(\Omega)$ the set of all distributions on $\Omega$.

If for every compact $K \subset \Omega$, the previous inequality holds for the same integer $N$, then the order of the distribution $u$ is at most $N$.

Example 1.1

(i) Every function $f \in L^1_{\text{loc}}(\Omega)$ defines a distribution $T_f \in \mathcal{D}'(\Omega)$ or order 0 by posing

$$\langle T_f, u \rangle = \int_\Omega f(x)u(x) \, dx, \quad \forall u \in \mathcal{D}(\Omega).$$

(ii) Given $a \in \Omega$, we consider the linear form on $\mathcal{D}(\Omega)$ defined as

$$\langle \delta_a, u \rangle = u(a), \quad \forall u \in \mathcal{D}(\Omega).$$

Hence, we have

$$|\langle \delta_a, u \rangle| \leq \|u\|_{L^\infty(\Omega)}, \quad \forall u \in \mathcal{D}(\Omega),$$

$\delta_a$ is a distribution of order 0 in $\Omega$, called the Dirac mass at point $a$. However, $\delta_a$ cannot be obtained via a $L^1_{\text{loc}}(\Omega)$ function. Indeed, suppose that there exists $f \in L^1_{\text{loc}}(\Omega)$ such that

$$\langle \delta_a, u \rangle = \int_\Omega f(x)u(x) \, dx = u(a), \quad \forall u \in \mathcal{D}(\Omega).$$

(1.9)

By denoting $\bar{\Omega} = \Omega \setminus \{a\}$, we have

$$\langle \delta_a, u \rangle = \int_{\bar{\Omega}} f(x)u(x) \, dx = 0, \quad \forall u \in \mathcal{D}(\Omega).$$

Hence, we can show that $f = 0$ almost everywhere in $\bar{\Omega}$ and thus almost everywhere in $\Omega$. Therefore, $\int_\Omega f(x)u(x) \, dx = 0$ for every function $u \in \mathcal{D}(\Omega)$. This is in contradiction with the assumption (1.9) when $u(a) \neq 0$.

The continuity property of a distribution can be characterized using sequences, as for the linear operators in normed spaces. We introduce first the notion of convergence in $\mathcal{D}(\Omega)$.

Definition 1.8 Let consider an open set $\Omega \subset \mathbb{R}^d$. The sequence $(u_n)_{n \in \mathbb{N}}$ of elements of $\mathcal{D}(\Omega)$ converges toward $u \in \mathcal{D}(\Omega)$ if

1. there exists a compact set $K$ in $\Omega$ such that $\text{Supp}(u_n) \subset K$, for all $n \in \mathbb{N}$.

2. for every $\alpha \in \mathbb{N}^d$, we have $\lim_{n \to \infty} \|\partial^\alpha (u_n - u)\|_{L^\infty} = 0$.

Definition 1.9 Let consider a sequence $(T_n)_{n \in \mathbb{N}}$ of elements in $\mathcal{D}'(\Omega)$ and a distribution $T$ on $\Omega$. The sequence $(T_n)_{n \in \mathbb{N}}$ is said to converge toward $T$ if and only if:

$$\forall u \in \mathcal{D}(\Omega), \quad \lim_{n \to \infty} \langle T_n, u \rangle = \langle T, u \rangle.$$

Theorem 1.9 A linear form $T : \mathcal{D}(\Omega) \to \mathbb{C}$ is a distribution on $\Omega$ if and only if, for any sequence $(u_n)_{n \in \mathbb{N}} \subset \mathcal{D}(\Omega)$ tending toward 0 in $\mathcal{D}(\Omega)$, we have:

$$\lim_{n \to \infty} \langle T, u_n \rangle = 0.$$
Proof. Suppose \( T \in \mathcal{D}'(\Omega) \) and consider \((u_n)_{n \in \mathbb{N}} \subset \mathcal{D}(\Omega)\) tending toward 0 in \( \mathcal{D}(\Omega) \). From the previous definition of the convergence in \( \mathcal{D}(\Omega) \), we know that there exists a compact subset \( K \) in \( \Omega \) such that \( \text{Supp}(u_n) \subset K \) for each \( n \in \mathbb{N} \). Hence, there exist constants \( C_K \) and \( N_K \) such that
\[
|\langle T, u_n \rangle| \leq C_K \sum_{|\alpha| \leq N_K} \|\partial^\alpha u_n\|_\infty, \quad \forall n \in \mathbb{N}.
\]
Since \( \lim_{n \to \infty} \|\partial^\alpha u_n\|_\infty = 0 \) for each given \( \alpha \), we deduce the result for every sequence \((u_n)_{n \in \mathbb{N}} \subset \mathcal{D}(\Omega)\) tending toward 0 in \( \mathcal{D}(\Omega) \). The reciprocal assertion is established by contradiction.

Suppose there exists a compact \( K_0 \) in \( \Omega \) such that for all \( N \in \mathbb{N} \) and for every \( C > 0 \) we could find \( u_{N,C} \in \mathcal{D}_{K_0}(\Omega) \) verifying
\[
|\langle T, u_{N,C} \rangle| > C \sum_{|\alpha| \leq N} \|\partial^\alpha u_{N,C}\|_\infty.
\]
Take \( u_N = u_{N,N} \) for every \( N \in \mathbb{N}^+ \). It is obvious that \( u_N \neq 0 \), hence assume that the following equality holds
\[
\sup_{|\alpha| \leq N} \|\partial^\alpha u_N\|_\infty = \frac{1}{N}.
\]
Hence, the two previous equalities yield the result
\[
|\langle T, u_N \rangle| > 1, \quad \forall N \in \mathbb{N}.
\]
Since, for every \( N \in \mathbb{N} \), \( \text{Supp}(u_N) \subset K_0 \) we deduce
\[
\lim_{N \to \infty} u_N = 0, \quad \text{in} \ \mathcal{D}(\Omega),
\]
that is obviously in contradiction with the hypothesis. \( \square \)

We may be wondering what is the meaning of the integral \( \int_\Omega f(x)u(x) \, dx \) for arbitrary functions and for every test function \( u \). Obviously, the function \( f \) must be in \( \tilde{L}^1(K) \) for every compact set \( K \) in \( \Omega \). The following theorem allows to consider functions in \( L^1_{\text{loc}}(\Omega) \) as distributions.

**Theorem 1.10** Consider a test function \( u \in \mathcal{D}_K(\Omega) \). The linear map \( \iota : L^1_{\text{loc}}(\Omega) \to \mathcal{D}'(\Omega), \ f \mapsto \iota(f) : u \to \int_\Omega f(x)u(x) \, dx \) is a linear injection. Furthermore, for any compact \( K \) in \( \Omega \), we have
\[
\forall u \in \mathcal{D}_K(\Omega), \quad |\iota(f), u| \leq \|f\|_{\tilde{L}^1(K)} \|u\|_{L^\infty(\Omega)}.
\]

**Remark 1.1** We will assume now that any function \( f \) in \( L^1_{\text{loc}}(\Omega) \) is identical to the distribution
\[
u \mapsto \int_\Omega f(x)u(x) \, dx.
\]
When considering that the distribution \( T \) is a function, this will mean that there exists a function \( f \in L^1_{\text{loc}}(\Omega) \) such that
\[
\forall u \in \mathcal{D}(\Omega), \quad \langle T, u \rangle = \int_\Omega f(x)u(x) \, dx.
\]
The following proposition establishes that the Dirac mass is the limit of the approximations of identity.

**Proposition 1.5** Consider a function \( \chi \in \mathcal{D}(\mathbb{R}^d) \) of integral 1 and \((\varepsilon_n)_{n \in \mathbb{N}} \) a sequence of real numbers tending toward 0. We pose
\[
\chi_{\varepsilon_n}(x) \overset{\text{def}}{=} \varepsilon_n^{-d} \chi \left( \frac{x}{\varepsilon_n} \right).
\]
Then, the sequence \((\chi_{\varepsilon_n})_{n \in \mathbb{N}} \) converges toward 0 in the distributional sense, where \( \delta_0 \) is the linear form defined by \( \delta_0 : \mathcal{D}(\mathbb{R}^d) \to \mathbb{C}, \ u \mapsto u(0) \).
Remark 1.2 This result has a physical meaning. Indeed, if the functions $\varepsilon_n$ are considered as mass densities, the limit (in a certain sense) must be seen as a punctual mass associated with the origin.

Definition 1.10 We call principal value of the function $1/x$ and we denote it $\text{pv} \frac{1}{x}$ the distribution defined by:

$$\langle \text{pv} \frac{1}{x}, u \rangle = \frac{1}{2} \int_{-\infty}^{\infty} \frac{u(x) - u(-x)}{x} \, dx, \quad \forall u \in \mathcal{D}(\mathbb{R}).$$

Notice that this formula defines a distribution since we have, for every test function $u \in \mathcal{D}[-R,R]$,

$$|\langle \text{pv} \frac{1}{x}, u \rangle| \leq R\|u\|_{L^\infty}.$$

1.2.2 Operations on distributions

We define now several usual operations on distributions that are already familiar for classical smooth functions, for example test functions.

Definition 1.11 (Restriction) Suppose $\omega$ and $\Omega$ are two open subsets in $\mathbb{R}^d$ such that $\omega \subset \Omega$ and consider a distribution $T$ on $\Omega$. We define the restriction of $T$ to the open set $\omega$, denoted by $T|_\omega$, as

$$\forall u \in \mathcal{D}(\omega), \quad \langle T|_\omega, u \rangle \overset{\text{def}}{=} \langle T, u \rangle.$$ 

Furthermore, if $f \in L^1_{\text{loc}}(\Omega)$ we have:

$$\forall u \in \mathcal{D}(\Omega), \quad \int_{\omega} f|_\omega(x)u(x) \, dx = \int_{\Omega} f|_\omega(x)u(x) \, dx.$$

This definition coincide with the notion of restriction for functions.

Remark 1.3 The restriction of a distribution to a set is only defined for an open set.

If $T_1$ and $T_2$ are distributions of order $m$ on $\mathbb{R}^d$, it is obvious that $T_1 + T_2$ and $aT_1$ are distributions of the same order for all $a \in \mathbb{R}$. However, the product $T_1T_2$ of two distributions is not a distribution. For example, if we were to define $\langle T_1T_2, u \rangle = \langle T_1, u \rangle \langle T_2, u \rangle$, this is not linear in $u$.

1.2.3 Differentiation of distributions

One of the strongest advantage of the distribution theory is that the differentiation operation is always defined and is "continuous". Indeed, consider a function $f \in C^1(\mathbb{R}^d)$ and $u \in \mathcal{D}(\mathbb{R}^d)$, then:

$$\left\langle \frac{\partial f}{\partial x_i}, u \right\rangle = \int_{\mathbb{R}^d} \frac{\partial f}{\partial x_i}(x)u(x) \, dx,$$

$$= -\int_{\mathbb{R}^d} \frac{\partial u}{\partial x_i}(x)f(x) \, dx = -\left\langle f, \frac{\partial u}{\partial x_i} \right\rangle.$$

This result is still valid if we set $f$ to be a distribution. More precisely,

Proposition 1.6 Let $T \in \mathcal{D}'(\Omega)$ be a distribution on an open set $\Omega \subset \mathbb{R}^d$. The linear form $T^{(j)}$ on $\mathcal{D}(\Omega)$ defined by:

$$\langle T^{(j)}, u \rangle \overset{\text{def}}{=} -\left\langle T, \frac{\partial u}{\partial x_j} \right\rangle = -\langle T, \partial_x u \rangle,$$

defines a distribution on $\Omega$. Furthermore, if $(T_n)_{n \in \mathbb{N}}$ is a sequence in $\mathcal{D}'(\Omega)$ converging toward $T$, then the sequence $(T_n^{(j)})_{n \in \mathbb{N}}$ converges toward $T^{(j)}$. 
Proof. Since \( T \) is a distribution, for every compact \( K \) in \( \Omega \), there exists a constant \( C \) and an integer \( N \) such that
\[
\forall v \in \mathcal{D}_K, \quad |\langle T, v \rangle| \leq C \sup_{|\alpha| \leq N} \|\partial^\alpha v\|_{L^\infty}.
\]
For every \( u \in \mathcal{D}_K \), we apply this to \( v = \partial_x^j u \in \mathcal{D}_K \). Hence, we have
\[
\forall u \in \mathcal{D}_K, \quad |\langle T^{(j)}, u \rangle| = |\langle T, \partial_x^j u \rangle| \leq C \sup_{|\alpha| \leq N+1} \|\partial^\alpha u\|_{L^\infty}.
\]
Thus, \( T^{(j)} \) is a distribution on \( \Omega \). Moreover, for every function \( u \in \mathcal{D}(\Omega) \) we have:
\[
\lim_{n \to \infty} \langle T^{(j)}_n, u \rangle = -\lim_{n \to \infty} \langle T_n, \partial_x^j u \rangle = -\langle T, \partial_x^j u \rangle = \langle T^{(j)}, u \rangle.
\]
the results follows. \( \square \)

Definition 1.12 Suppose \( T \in \mathcal{D}'(\Omega) \). Then, the distribution \( T^{(j)} \) is called the partial derivative of \( T \) with respect to the variable \( x_j \) and is denoted \( \partial_x^j T \) and is such that:
\[
\langle \partial_x^j T, u \rangle = -\langle T, \partial_x^j u \rangle, \quad \forall u \in \mathcal{D}(\Omega).
\]

Notice that if \( f \) is a differentiable function, the derivative of the distribution associated with \( f \) coincide with the usual derivative of \( f \).

Example 1.2 (i) The Heaviside step function \( H \) is the characteristic function of \( \mathbb{R}^+ \). We have \( \partial_x H = \delta_0 \) in \( \mathcal{D}'(\mathbb{R}) \). Indeed, for any test function \( u \in \mathcal{D}(\mathbb{R}) \), we have:
\[
\langle \partial_x H, u \rangle = \left\langle \frac{dH}{dx}, u \right\rangle = -\langle H, u' \rangle = -\int_{-\infty}^{+\infty} u'(x)H(x) \, dx
\]
\[
= -\int_{0}^{+\infty} u'(x) \, dx = u(0) \quad \text{since} \quad u(+\infty) = 0
\]
\[
= \langle \delta_0, u \rangle.
\]
Hence, the derivative of \( H \) in the sens of distributions is the Dirac mass defined hereabove.

(ii) Since the function \( f(x) = \log |x| \) is in the space \( L^1_{\text{loc}}(\mathbb{R}) \), it defines an element of \( \mathcal{D}'(\mathbb{R}) \). Indeed, the derivative of \( f \) in the distributional sense is the distribution \( \text{pv} \frac{1}{x} \) introduced previously:
\[
\frac{df}{dx} = \text{pv} \frac{1}{x}.
\]
According to the definition of the derivative in \( \mathcal{D}'(\mathbb{R}) \), we have for all \( u \in \mathcal{D}(\mathbb{R}) \):
\[
\left\langle \frac{df}{dx}, u \right\rangle = -\int_{\mathbb{R}} \log(|x|)u'(x) \, dx.
\]
We observe that:
\[
\int_{|x| > \varepsilon} \log(|x|)u'(x) \, dx = [u(-\varepsilon) - u(\varepsilon)] \log \varepsilon - \int_{|x| > \varepsilon} u'(x) \, dx,
\]
and thus, we deduce that
\[
\int_{\mathbb{R}} \log(|x|)u'(x) \, dx = \lim_{n \to \infty} \int_{|x| > \varepsilon} \log(|x|)u'(x) \, dx = -\langle \text{pv} \frac{1}{x}, u \rangle.
\]
The next results establishes a relation between the derivative and the primitive, in the distributional sense.

**Theorem 1.11** Consider a distribution \( T \) on an open interval \( I \subset \mathbb{R} \). If its derivative, in the distributional sense, \( T' \) is a function of \( L^1_{loc}(I) \), then \( T \) is a continuous function and we have, for all \( a \in I \):

\[
T(x) = \int_a^x T'(x)dx + C.
\]

**Lemma 1.6** Suppose \( I \) is an open interval of \( \mathbb{R} \). Then,

1. the distributions on \( I \) for which the identity \( T' = 0 \) holds, are the constant functions.
2. for any \( T_2 \in \mathcal{D}'(I) \), there exists \( T_1 \in \mathcal{D}'(I) \) such that \( T_1' = T_2 \).
3. for \( a \in I \) and \( f \in L^1_{loc}(I) \), we have (in the distributional sense):

\[
\frac{d}{dx} \int_a^x f(y)dy = f(x).
\]

**Proof.** Let consider \( \theta \in \mathcal{D}(I) \), such that \( \int_I \theta(x)dx = 1 \), and \( u \in \mathcal{D}(I) \). Then, the function

\[
\eta(x) = u(x) - \left( \int_I u(x)dx \right) \theta(x) \in \mathcal{D}(I) \quad \text{and} \quad \int_I \eta(x)dx = 0.
\]

This implies the existence of a unique function \( v \in \mathcal{D}(I) \) such that \( v' = \eta \). Hence, there exists a unique function \( v \in \mathcal{D}(I) \) such that

\[
v' = u - \left( \int_I u(x)dx \right) \theta \tag{1.12}
\]

Now, let consider \( T \in \mathcal{D}'(I) \) such that \( T' = 0 \). We have

\[
\langle T, u \rangle = \left( \int_I u(x)dx \right) \langle T', \theta \rangle + \langle T, v' \rangle.
\]

The term \( \langle T, v' \rangle = -\langle T', v \rangle \) is equal to zero and we obtain by denoting \( C \) the constant term \( \langle T, \theta \rangle \):

\[
\langle T, u \rangle = C \int_I u(x)dx.
\]

Thus, \( T \) is equal to a constant \( C \).

To find a primitive \( T_1 \) of \( T_2 \), we pose

\[
\langle T_1, u \rangle = -\langle T_2, v \rangle,
\]

where \( v \) is the unique function of \( \mathcal{D}(I) \) associated with \( u \) via the relation (1.12). It is easy to show the linearity and continuity of \( T_1 \) and thus we have

\[
\langle T_1, u' \rangle = -\langle T_2, u \rangle, \quad \forall u \in \mathcal{D}(I),
\]

and thus we have proved that \( T_1' = T_2 \).

**Lemma 1.7** Consider \( T \in \mathcal{D}'(\mathbb{R}^d) \) and suppose that \( \partial_{x_j} T = 0 \) for all \( i = 1, \ldots, d \). Then, \( T \) is a constant (function).
1.2.4 Multiplication by a $C^\infty$ function

**Proposition 1.7** Let consider $f \in C^\infty(\Omega)$ and $T \in \mathcal{D}'(\Omega)$. Then, for all $u \in \mathcal{D}(\Omega)$, the linear form on $\mathcal{D}(\Omega)$ defined by $u \mapsto \langle T, fu \rangle$ is a distribution on $\Omega$. The order of this distribution on every compact $K$ in $\Omega$ in lesser than or equal to the order of $T$ on $K$.

**Definition 1.13** The product of the distribution $T \in \mathcal{D}'(\Omega)$ by the function $f \in C^\infty(\Omega)$ is the distribution defined by:

$$\langle fT, u \rangle = \langle T, fu \rangle, \quad \forall u \in \mathcal{D}(\Omega).$$

**Example 1.3**

(i) For $f \in C^\infty(\Omega)$ and $a \in \Omega$, we have $f\delta_a = f(a)\delta_a$. Indeed, we can observe that

$$\langle f\delta_a, u \rangle = \langle \delta_a, fu \rangle = \langle f(a)\delta_a, u \rangle, \quad \forall u \in \mathcal{D}(\Omega).$$

In particular, $x\delta_0 = 0$.

(ii) We can find all solutions $T \in \mathcal{D}'(\Omega)$ of the equation $xT = 0$. Let consider $\chi \in \mathcal{D}(\mathbb{R})$ such that $\chi(0) = 1$. For every function $u \in \mathcal{D}(\mathbb{R})$, the function $u - u(0)\chi$ vanishes at the origin, hence there exists a function $v \in C^\infty(\mathbb{R})$ such that

$$u = u(0)\chi + xv.$$ 

Since $u$ and $\chi$ are functions with compact support, the relation means that $v \in \mathcal{D}(\mathbb{R})$. Thus, we have

$$\langle T, u \rangle = u(0)\langle T, \chi \rangle + \langle T, xv \rangle.$$ 

By assumption, we have $\langle T, xv \rangle = \langle xT, v \rangle = 0$. Denoting $C$ the constant $\langle T, \chi \rangle$, we obtain that $\langle T, u \rangle = Cu(0)$, hence $T = C\delta_0$.

**Proposition 1.8 (Leibniz formula)** Consider $f \in C^\infty(\Omega)$ and $T \in \mathcal{D}'(\Omega)$. Then,

$$\partial_{x_i}(fT) = \frac{\partial f}{\partial x_i}T + f\partial_{x_i}T, \quad \forall i \in \{1, \ldots, n\}.$$