

The finite difference method

In this numerical experiment, we will use Scilab to compute an approximation of the solution $u : [0, 1] \rightarrow \mathbb{R}$ of the following Dirichlet boundary-value problem:

$$\begin{cases}
-u''(x) + b(x)u'(x) + c(x)u(x) = f(x),
\quad u(0) = a_0, \ u(1) = a_1.
\end{cases} \quad (1)$$

Here, the functions $f$, $b$ and $c$ are given and are supposed such that there exists a unique solution to this problem. Furthermore, we assume the solution $u$ to be sufficiently smooth.

**Exercice 1 : Discretization of the problem** ($\mathcal{D}$)

For $N \in \mathbb{N}^*$, we pose $h = 1/(N + 1)$ and we define the points

$$x_k = kh, \quad k = 1, \ldots, N.$$ 

To alleviate the notations, we will denote by $b_k = b(x_k)$, $c_k = c(x_k)$ and $f_k = f(x_k)$.

1. We note $u_0 = u(0) = a_0$, $u_{N+1} = u(1) = a_1$ and for every $k = 1, \ldots, N$, $u_k$ represents an approximation (that will be determined in the experiment) of $u(x_k)$.

Explain and justify the following finite difference scheme for the problem ($\mathcal{D}$):

$$-u_{k-1} + 2u_k - u_{k+1} = \frac{b_k u_{k+1} - u_{k-1}}{2h} + c_k u_k = f_k, \quad k = 1, \ldots, N. \quad (2)$$

2. We pose $u_k = (u_1, \ldots, u_N)^t$. Show that the previous relations can be written in a more compact form as:

$$A_h u_h = v_h, \quad (3)$$

where $A_h \in \mathcal{M}_N(\mathbb{R})$ and $v_h \in \mathbb{R}^N$ must be determined.

**Exercice 2 : Resolution of the linear system** ($\mathcal{D}$)

In this section, we assume that both functions $b$ and $c$ are identically equal to zero.

1. Show that for every vector $x = (x_1, \ldots, x_N)^t \in \mathbb{R}^N$, the following identity holds:

$$\langle A_h x, x \rangle = \frac{1}{h^2} \left[ x_1^2 + x_N^2 + \sum_{k=2}^{N} (x_k - x_{k-1})^2 \right].$$

Deduce that the matrix $A_h$ is invertible.

2. Computation of the stiffness matrix $A_h$ and of the right-hand term $v_h$ of the system ($\mathcal{D}$).
(a) Write a Scilab function to compute the stiffness matrix $A_h$. The header of this function shall be

```plaintext
function A=CalculMat(N)
```

Input argument : an integer $N$.
Output argument : a square array $A$ containing the coefficients of $A_h$.

(b) Write a function to construct the vector $v_h$. The header of this function shall be

```plaintext
function v=CalculSM(N,f,a0,a1)
```

Input argument : the number of points $N$, $f$ the name of the function computing the right-hand side term $f$, $a0$ and $a1$ the two Dirichlet boundary conditions $u(0) = a_0$ and $u(1) = a_1$.
Output argument : $v$ containing the vector $v_h$ from the identity (3).

3. Validation. Now, we suppose the function $f$ to be constant and equal to 1, and we impose homogeneous boundary conditions $a_0 = a_1 = 0$.

(a) Determine, by hand, a solution $u^c$ for the problem (D).

(b) Write a function for constructing the vector $u_h^c = (u^c(x_1), \ldots, u^c(x_N))^t$. The header of this function shall be

```plaintext
function ue=SolExa0(N)
```

Input argument : $N$.
Output argument : $ue$ containing the vector $u_h^c$.

(c) Solve the system (3) using Scilab. Compare the vectors $u_h$ and $u_h^c$, by computing the norm 2 (or any other equivalent norm) of their difference. Please comment.


(a) Determine, by hand, a function $f$ such that $u^c(x) = e^{-4x} \sin(\pi x)$ is a solution of the problem (D). What are the values of $a_0$ and $a_1$?

(b) We like to investigate the convergence of the finite difference method based on the analysis of the decrease of the error $u_h - u_h^c$ with respect to the parameter $N$. We remind that the method is said to be of order $p \in \mathbb{R}^+$ if there exists a real value $C$ such that, asymptotically,

$$
\|u_h - u_h^c\| \leq \frac{C}{N^{p}}.
$$

The order $p$ will depend on the selected norm. For different values of $N$ (for example, $n = 6 \times 2^j, j = 1, \ldots, 7$), compute the relative errors $\|u_h - u_h^c\|_1$, $\|u_h - u_h^c\|_2$ and $\|u_h - u_h^c\|_\infty$. Represent on the same plot the three curves $\|u_h - u_h^c\|_q$ with respect to $N$ in log–log scale. Find $p$. Please comment.

**Exercice 3 : Eigenvalues of the matrix $A_h$, condition number**

We suppose here that the function $b$ is identically null and that the function $c$ is constant.

1. Show, by hand, that the $N$ eigenvalues of the matrix $A_h$ are

$$
\lambda_{N,k} = c + \frac{4}{h^2} \sin^2\left(\frac{k \pi h}{2}\right), \quad k = 1, \ldots, N,
$$
associated with the $N$ eigenvectors

$$\varphi_{N,k} = (\sin(k\pi h), \sin(2k\pi h), \ldots, \sin(Nk\pi h))^t, \quad k = 1, \ldots, N.$$  

2. Compute, by hand, the eigenvalues of the operator $u \mapsto -u'' + cu$ endowed with the homogeneous boundary conditions; i.e. the scalars $\lambda$ for which there exists a non zero function $\psi$ (the so-called the eigenfunction associated with $\lambda$) verifying $\psi(0) = \psi(1) = 0$ and such that $-\psi''(x) + c\psi(x) = \lambda\psi(x)$.

3. Compare (by hand) the eigenvalues of the matrix $A_h$ with the eigenvalues of the operator $u \mapsto -u'' + cu$ computed previously.

4. For different values of $N$, represent on the same plot the eigenvalues of $A_h$ and that of the operator $u \mapsto -u'' + u$. Please comment.

5. We can define the condition number of the matrix $A_h$ in any given norm, as :

$$\text{cond}(A_h) = \|A_h\| \|A_h^{-1}\|.$$  

For $c = 0$, represent graphically, in log–log scale, the conditioning in norm 2 of the matrix $A_h$ with respect to $N$. Consider for instance $N = 6 \times 2^j, \ j = 1, \ldots, 7$. Please comment.

**Exercice 4 : An ill-posed problem**

We suppose here that $c = -\pi^2$. For different values of $N = 50, 100, 150, 200, \ldots$ solve the linear system (3) with a constant right-hand side term, equal to 1. Note each time the minimal value attained by the solution computed using Scilab. Please comment the results.

**Exercice 5 : The general case**

We suppose now that $b(x)$ and $c(x)$ are non constant functions and that $a_0$ and $a_1$ are non null. Write Scilab functions to compute $b(x) = \bar{b}\cos(\pi x)$ and $c(x) = \bar{c}\sin(\pi x)$, with $\bar{b}$ and $\bar{c}$ two constants, a function

$$\text{function } A=\text{CalculMatBC}(N)$$

to compute the matrix and a function for the right-hand side term corresponding to the exact solution $u_e(x) = e^{-4x}\cos(\pi x)$. Write a script to compare graphically the finite difference solution with the exact solution computed at the points $x_i$. Change the coefficients $\bar{b}$ and $\bar{c}$ so as to illustrate the second stability criterion

$$\max |b(x)|h \leq \frac{1}{2}.$$