Streamline Topologies in Stokes Flow
Within Lid-Driven Cavities

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Abstract. Stokes flow in a rectangular cavity with two moving lids (with equal speed but in opposite directions) and aspect ratio $A$ (height to width) is considered. An analytic solution for the streamfunction, $\psi$, expressed as an infinite series of Papkovich–Fadle eigenfunctions is used to reveal changes in flow structures as $A$ is varied. Reducing $A$ from $A = 0.9$ produces a sequence of flow transformations at which a saddle stagnation point changes to a centre (or vice versa) with the generation of two additional stagnation points. To obtain the local flow topology as $A \to 0$, we expand the velocity field about the centre of the cavity and then use topological methods. Expansion coefficients depend on the cavity aspect ratio which is considered as a separation parameter. The normal-form transformations result in a much simplified system of differential equations for the streamlines encapsulating all features of the original system. Using the simplified system, streamline patterns and their bifurcations are obtained, as $A \to 0$.

1. Introduction

Two-dimensional Stokes flow within a rectangular cavity, induced by the motion of the upper and lower lids with equal speed but in opposite directions, is investigated using both an analytic solution for the streamfunction and topological methods for the velocity field. The cavity aspect ratio $A$ (height to width) is considered as a separation parameter.

The cavity flow for a single moving lid and varying $A$ has been used extensively as a benchmark model for testing various numerical solution methods (Burggraf, 1966; Pan and Acrivos 1967; Gatski et al., 1982, 1988). Using Smith’s (1952) biorthogonality technique the analytical solution in which the streamfunction is expressed as a series of Papkovich–Fadle eigenfunctions, was first obtained by Joseph and Sturges (1978) who showed the presence of four eddies with corner eddies for an aspect ratio $A = 5$. Shankar (1993) considered the same problem and used a least-squares method rather than a biorthogonality condition to determine the coefficients in the infinite series. He compared his results with those of Joseph and Sturges (1978), and concluded that his method provides results as accurate as those via the biorthogonality condition. He also showed how the corner eddies grow, merge and produce a new eddy as $A$ increases. This is a mechanism for eddy generation that was previously described by Shen and Floryan (1985) for the flow over a cavity.

Subsequently, Sturges (1986) considered the symmetric flow with the lids moving in opposite directions (i.e. speed ratio $S = V_1/V_2 = -1$ where $V_1$ and $V_2$ are the upper and lower lid velocities, respectively). For four values of the aspect ratio he showed four distinct flow structures which revealed the presence of side ed-
dies attached to the stationary side walls. As $A$ was increased, these side eddies were found to grow in size and merge to establish a new eddy. This is a second mechanism for eddy generation in a lid-driven rectangular cavity. Shankar and Deshpande (2000) reviewed circulation flows in the cavity and gave a detailed history for the lid-driven cavity flow.

For $A \geq 0.9$ and $S = -1$ previous work has fully described the structure of the flow and a mechanism for eddy generation as $A$ is increased. However, the flow structure for smaller aspect ratios has not been examined. As in the problem of the flow in a cavity with free surfaces (Gaskell et al., 1998) and the flow in a half-filled annulus between rotating co-axial cylinders (Gaskell et al., 1997) we shall determine flow transformations as $A$ decreases and examine how the flow structure is transformed as a stagnation point changes from a centre to a saddle and vice versa.

As $A \to 0$, the local streamline patterns about the centre of the cavity are investigated by expanding the velocity field in a Taylor series and then using methods from dynamical systems. The qualitative study of streamlines using tools and ideas from dynamical systems has a long and scattered history in fluid mechanics. Dean (1950) considered a Taylor expansion as a solution to the Navier–Stokes equations in the vicinity of a planar wall. Oswatitsch (1958) and Legendre (1956) considered the topology in the vicinity of points of separation and attachment on a planar wall. Reviews on the flow topology are given by Tobak and Peake (1982) and Perry and Chong (1987).

The topological changes in the structure of the flow occur when stagnation points are created or destroyed or undergo a change of type (e.g. a centre becoming a saddle). The term bifurcation refers to such a change in the stagnation point configuration as the parameter is changed. However, note that in the present study, for each value of the parameter $A$, there is always a unique stable steady velocity field $\mathbf{v} = \mathbf{v}(x, A)$, but as the parameter is varied bifurcations in streamline patterns (i.e. for the ordinary differential equations $\dot{x} = \mathbf{v}$) occur. It is the bifurcations of these equations considered as a dynamical system that is explored in Section 3. Several authors have previously used this idea, for example, bifurcations near a critical point on a stationary plane and in a three-dimensional flow pattern were studied by Bakker (1989) and Dallmann (1988), respectively. Brøns and Hartnack (1999) and Hartnack (1999) considered flow bifurcations in a two-dimensional incompressible flow using normal-form transformations. Their approach is used here to reduce the number of terms in the system of differential equations for streamlines. From this, we obtain bifurcations close to a simple linear degeneracy at the centre of the cavity.

### 2. Governing Equation and Series Solution

Flow in a lid-driven rectangular cavity with two stationary walls and two moving lids is governed by the Navier–Stokes equation. It is assumed that the fluid is Newtonian and incompressible with density $\rho$ and viscosity $\mu$. The flow is steady and two-dimensional in the $(X, Y)$ plane with velocity $\mathbf{v} = (U, V)$. Body forces and inertia effects are negligible and so the flow is governed by Stokes equations.

With non-dimensional variables, $x = X/L$, $y = Y/L$, $u = U/U_1$ and $v = V/U_1$, the velocity components are expressed in terms of a streamfunction $\psi$:

$$
u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \quad (1)$$

where $\psi$ is a solution of the biharmonic equation

$$\nabla^4 \psi = 0. \quad (2)$$

The streamfunction is zero on the boundaries

$$\psi(\pm 1, y) = \psi(x, \pm A) = 0. \quad (3)$$

Using the relations (1) the no-slip conditions for the upper and lower lids and two stationary walls can be written in terms of derivatives of the streamfunction

$$\frac{\partial \psi}{\partial y}(x, A) = -1, \quad \frac{\partial \psi}{\partial y}(x, -A) = 1, \quad \frac{\partial \psi}{\partial x}(\pm 1, y) = 0. \quad (4)$$
The problem is reduced to the boundary value problem shown in Figure 1. Following Joseph and Sturges (1978) and Sturges (1986) the solution of the symmetric problem is of the form

\[ \psi(x, y) = \sum_{n=-\infty}^{\infty} A_n e^{s_n(y-A)} + e^{-s_n(y+A)} s_n^2 \varphi_n(x), \]  

where the functions \( \varphi_n = s_n \sin s_n \cos s_n x - s_n \cos s_n \sin s_n x \) are even Papkovich–Fadle eigenfunctions satisfying the no-slip boundary conditions on the side walls. The parameters \( s_n \) are complex eigenvalues determined by the side wall conditions \( \psi = \partial \psi / \partial x = 0 \) at \( x = \pm 1 \), which yield the eigenvalue equation

\[ \sin 2s_n = -2s_n. \]  

(6)

The \( s_n \) may easily be determined by the simple Newton iteration procedure described by Robbins and Smith (1948) using an initial estimate of the form

\[ 2s_n \approx (2n - 1.5)\pi + i \log(4n - 1)\pi. \]

It is easy to show from (6) that \( s_n = \overline{s_{-n}} \) (Joseph and Sturges, 1978) and using this yields \( \varphi^{-n}(x) = \overline{\varphi^n(x)} \) where the overbar denotes complex conjugate. Since the streamfunction is real, this implies \( A_{-n} = \overline{A_n} \). The coefficient \( A_n \) are determined from the boundary conditions using a truncation technique employing Smith’s biorthogonality relation which yields \( 2N \) equations for the \( 2N \) unknowns, i.e. the real and imaginary parts of \( A_n \) for \( n = 1, 2, \ldots, N \), which cannot be solved analytically. It is found computationally that for all aspect ratios investigated, \( A \in [0.05, 3] \), as \( n \) increases \( A_n = O(1/n^2) \) and hence this procedure converges because of the strong influence of the exponential factor in the solution (5). When the coefficients have been determined, the streamfunction at any interior point in the liquid is obtained by simply summing a finite number of terms in the series (5). Most of the figures in this paper were produced using a truncation number of \( N = 200 \).

### 2.1. Streamlines Determined by the Analytic Solution

**Case 1: \( A \geq 0.9 \)** For \( A = 0.9 \) Figure 2(a) shows streamlines for a single eddy with a centre at \((0, 0)\). As might be expected, the flow pattern is symmetric about \( x = 0, y = 0 \). As \( A \) is gradually increased, a sequence of flow transformations unfold by which two additional eddies are generated in the cavity. For example, a flow transformation arises at \( A_{c1} \approx 0.931 \) (to three decimal places) when the centre at \((0, 0)\) becomes a saddle point, see Figure 2(b) where \( A = 1.5 \). The second critical aspect ratio is \( A_{c2} \approx 2.498 \) at which two degenerate critical points appear on the two side walls where side eddies (separation bubbles) are about to emerge once \( A \) is increased. The genesis of these critical points and the development of separation bubbles was considered by Bakker (1989). Figure 2(c) where \( A = 2.75 \), shows a saddle point at the origin with a separatrix, two sub-eddies and two isolated side eddies. The latter are located adjacent to each stationary side wall and are bounded by a heteroclinic connection between two saddle points having the streamfunction value, \( \psi = 0 \). As \( A \) gradually increases the side eddies expand and approach the saddle point at the origin and at \( A_{c3} \approx 2.789 \) the heteroclinic connections merge with each other at the saddle point producing four heteroclinic connections between the saddle point and the four separation/saddle points on the side walls as shown.
in Figure 2(d). This is a global bifurcation, involving the formation of two saddle-point triangles as a result of the various saddle saddle ss connections. Once again Bakker gave a general description of a saddle-point-triangle bifurcation and showed that the ss connections are straight lines if terms of order four and above are ignored. At this critical aspect ratio, $A_c$, there are now two complete eddies within the cavity and between them a third is about to be born. Indeed, as $A$ is increased beyond $A_c$ the heteroclinic connections from the four separation/saddle points separate from the saddle point at $x = 0$ (see Figure 2(e) with $A = 2.8$). There are now two heteroclinic connections crossing the cavity, each connected to separation/saddle points on the side walls – ($P_1$, $P_3$) and ($P_2$, $P_4$) in Figure 2(e) – and between these two connections lies the saddle point at the origin with its separatrix enclosing two sub-eddies. As $A$ increases, the sub-eddy centres lying on the line $y = 0$ approach the saddle point at the origin, Figure 2(f), and coalesce at $A_c \approx 2.910$ to produce a centre. At this critical aspect ratio the development of the third eddy, between the other two, is complete so that three eddies now occupy the cavity, Figure 2(g). Figure 2 shows the eddy generation in which a single eddy develops into three as $A$ is increased from 0.9 to 3.0.

**Case 2: $A < 0.9$.** When the aspect ratio is reduced below $A = 0.9$, again there is a flow transformation with the centre at $(0, 0)$ becoming a saddle point. $A_{c_1} \approx 0.318$ is the critical aspect ratio below which the origin is a saddle point as illustrated in Figure 3(a) for which $A = 0.25$. This figure consists of a saddle point on $x = 0$ with a separatrix and two centres.

There are critical values of the aspect ratio where the flow structure is consequently changed, see Table 1, as $A$ is decreased. At critical aspect ratios $A_{c_1} \approx 0.318$, $A_{c_2} \approx 0.169$, $A_{c_3} \approx 0.115$, etc. the centre alternately changes to a saddle point and then a centre and so on. Each successive separation on $x = 0$ generates two additional stagnation points. Their positions can be determined using the bisection method to find where $u = v = 0$. For aspect ratio $A = 0.15$, which lies between $A_{c_2}$ and $A_{c_3}$, the flow pattern is shown in Figure 3(b), where it is seen to consist of a centre on $x = 0$, with two other centres plus two saddle points and a separatrix; see Figure 3(c), which shows an enlargement of the separatrix in Figure 3(b) for the sake of clarity. For $A \approx 0.115$ seven stagnation points appear on $y = 0$. There are several possible flow structures which feature seven stagnation points and examining the values of $\psi$ and its discriminant at these points should reveal the actual flow pattern. However, a calculation shows that streamfunction values at these points are too
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Figure 3. Streamlines for the symmetric lid-driven cavity with (a) $A = 0.25$ and (b) $A = 0.15$. (c) The separatrix in (b) has been enlarged to reveal its structure.

Table 1. Critical values of $A$ and corresponding stagnation-point separations.

<table>
<thead>
<tr>
<th>Aspect ratio</th>
<th>Separation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{c_1} \cong 0.318$</td>
<td>Centre $\rightarrow$ Saddle</td>
</tr>
<tr>
<td>$A_{c_2} \cong 0.169$</td>
<td>Saddle $\rightarrow$ Centre</td>
</tr>
<tr>
<td>$A_{c_3} \cong 0.115$</td>
<td>Centre $\rightarrow$ Saddle</td>
</tr>
</tbody>
</table>

close to each other (differing in the tenth or eleventh decimal place), and hence the streamlines about the origin cannot be determined explicitly. In the next section we use a dynamical systems approach to obtain the local flow structure near the origin in the cavity.

3. Local Streamline Topology

For the symmetric cavity flow, the streamline patterns in the neighbourhood of $(0, 0)$ can be investigated via the velocity vector field $\dot{x} = u = \partial\psi/\partial y$ and $\dot{y} = v = -\partial\psi/\partial x$, where $\psi$ is a Taylor expansion of expression (5) about $(0, 0)$ given by

$$
\psi = \sum_{i+j=0}^{\infty} c_{2i,2j} x^{2i} y^{2j},
$$

and each coefficient $c_{2i,2j}$ is given as an infinite series

$$
c_{2i,2j} = \sum_{n=-N}^{N} \left\{ (-1)^{i} \frac{s_{n}^{2i-1}}{(2i)!} \left[ s_{n} \sin s_{n} + 2i \cos s_{n} \right] \left( \frac{s_{n}^{2j}}{(2j)!} e^{-s_{n}A(2A_{n})} \right) \right\},
$$

for $i = 0, 1, 2, \ldots$, $j = 0, 1, 2, \ldots$, and $N$ is a truncation number. The expansion coefficients in expression (7) have direct physical interpretations. The first-order coefficients are related to the components of the velocity vectors at the origin, and
therefore vanish. The higher-order coefficients are related to the viscous stress tensor, which is given by
\[ \tau = \mu \left( \frac{\partial u}{\partial y} \right)_{y=0} = \mu 2c_{0,2}, \quad (9) \]
where \( \mu \) is the dynamic viscosity of the fluid. The linear approximation of the equations for the streamlines is
\[ \left( \begin{array}{c} \dot{x} \\ \dot{y} \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right) + \left( \begin{array}{cc} 0 & 2c_{0,2} \\ -2c_{2,0} & 0 \end{array} \right) \left( \begin{array}{c} x \\ y \end{array} \right). \quad (10) \]

It is clear that the origin is always a critical point and the local flow pattern is determined using the standard theory for Hamiltonian systems. If the determinant of the Jacobian matrix \( \det(J) = 4c_{2,0}c_{0,2} \) is positive the critical point is a centre and if \( \det(J) \) is negative it is referred to as a saddle point.

We now turn to the cavity flow. For varying \( A \) each coefficient in (7) can easily be calculated using (8). As \( A \) is decreased from 0.9 to 0, \( c_{0,2} \) is always positive whereas \( c_{2,0} \) changes sign at critical values, \( A_{c_1}, A_{c_2}, \ldots \), see Table 1. For example, for \( A \in (0.9, 0.318) \), \( \det(J) = 4c_{2,0}c_{0,2} > 0 \) and the origin is a centre. When \( c_{2,0} = 0 \) (\( \det(J) = 0 \)) for \( A \approx 0.318 \) the stagnation point is degenerate and higher-order terms become decisive for the streamline pattern. For a significant simplification of higher-order terms in the streamfunction and an easy determination of simple degenerate patterns and their bifurcations, normal-form transformations are used.

### 3.1. The Normal-Form Transformation

The system under consideration is Hamiltonian and, to preserve this property, canonical transformations have been widely used in analytical mechanics (see Brøns and Hartnack, 1999; Hartnack, 1999). Transformations conserving the symmetry are found via generating functions (see Goldstein, 1950). In order to show how to simplify cubic terms of the system – fourth-order terms of \( \psi \) – we choose a generating function
\[ G = \xi y + s_{3,0}\xi^3 + s_{0,3}y^3 + s_{3,1}\xi^3y + s_{1,3}\xi y^3, \quad (11) \]

which defines a canonical transformation through
\[ x = \frac{\partial G}{\partial \eta}, \quad \eta = \frac{\partial G}{\partial \xi}, \quad (12) \]
where \( (\xi, \eta) \) are new coordinates. This and (11) become
\[ x = \xi + 3s_{0,3}y^2 + s_{3,1}\xi^3 + 3s_{1,3}\xi y^2, \]
\[ \eta = y + 3s_{3,0}\xi^2 + 3s_{3,1}\xi^2 y + s_{1,3}y^3. \quad (13) \]

Equation (13) can be solved in the form of Taylor series,
\[ x = \xi + \sum_{i+j=2}^{\infty} \alpha_{i,j}\xi^i\eta^j \quad \text{and} \quad y = \eta + \sum_{i+j=2}^{\infty} \beta_{i,j}\xi^i\eta^j. \quad (14) \]

By inserting these in (13) and collecting terms of the same order in \( \xi, \eta \), we obtain the linear equations for the Taylor coefficients \( \alpha_{i,j}, \beta_{i,j} \). For the third-order coefficients the solutions are
\[ \alpha_{1,1} = \alpha_{2,0} = \alpha_{0,3} = 0, \quad \alpha_{0,2} = 3s_{0,3}, \quad \alpha_{3,0} = s_{3,1}, \quad \alpha_{1,2} = 3s_{1,3}, \quad \alpha_{2,1} = -18s_{0,3}s_{3,0}, \]
\[ \beta_{1,1} = \beta_{0,2} = \beta_{3,0} = \beta_{1,2} = 0, \quad \beta_{2,0} = -3s_{3,0}, \quad \beta_{0,3} = -s_{1,3}, \quad \beta_{2,1} = -3s_{3,1}. \]

Thus the transformation (14), up to third-order terms, leads to
\[ x = \xi + 3s_{0,3}\eta^2 + 3s_{1,3}\xi \eta^2 + s_{3,1}\xi^3 - 18s_{0,3}s_{3,0}\xi^2\eta. \quad (15) \]
Inserting this transformation in (7) we obtain
\[
\psi = c_{0,0} + c_{0,2} \eta^2 - (6c_{0,2}s_{3,0}) \xi^2 \eta + (c_{2,2} - 6c_{0,2}s_{3,1}) \xi^2 \eta^2 \\
+ (c_{4,0} - 9c_{0,2}s_{3,0}) \xi^4 + (c_{0,4} - 2c_{0,2}s_{1,3}) \eta^4 + O((\xi, \eta)^6).
\]  
(17)

The \( s_{i,j} \) are free for us to choose; \( s_{0,3} \) does not appear in (17) and it can be arbitrarily set to zero. By choosing
\[
s_{3,0} = 0, \quad s_{1,3} = \frac{c_{0,4}}{2c_{0,2}}, \quad s_{3,1} = \frac{c_{2,2}}{6c_{0,2}},
\]
(18)
a number of terms are eliminated and thereby one obtains
\[
\psi = c_{0,0} + c_{0,2} \eta^2 + c_{4,0} \xi^4 + O((\xi, \eta)^6).
\]  
(19)

If \( c_{4,0} \neq 0 \), the only remaining fourth-order term of \( \psi \) is non-degenerate, and the local streamline pattern can be determined. If \( c_{4,0} = 0 \), the critical point is degenerate to fourth order, and higher-order terms in the expansion of \( \psi \) must be computed. To eliminate terms of sixth order in \( \psi \), we use a generating function of the form
\[
G = \xi \psi + s_{3,0} \xi^3 + s_{3,1} \xi^3 \psi + s_{5,0} \xi^5 + s_{5,1} \xi^5 \psi + s_{3,3} \xi^3 \psi^3 + s_{1,5} \xi^5 \psi^5,
\]
(20)

where the \( s_{i,j} \) of order four are given by (18) and the \( s_{i,j} \) of order five and six are now free parameters. It can be shown that by carrying on in this manner, keeping track of higher-order terms, in the general case to any finite order \( 2N \) one obtains
\[
\psi = c_{0,0} + c_{0,2} \eta^2 + c_{2N,0} \xi^{2N} + O((\xi, \eta)^{2N+2}).
\]  
(21)

Dropping the \( O \)-terms in (21) gives the normal-form of order \( 2N \) for the streamfunction for a linear degenerate critical point. From the normal-form the local flow topology in the neighbourhood of the degenerate critical point can easily be obtained. A degeneracy of order \( 2N \) occurs if \( c_{4,0} = c_{6,0} = \cdots = c_{2N-2,0} = 0 \) but \( c_{2N,0} \neq 0 \). The normal-form of \( \psi \) is
\[
\psi = c_{0,0} + c_{0,2} \eta^2 + c_{2N,0} \xi^{2N}.
\]  
(22)

Since streamlines lie on isocurves of \( \psi \) we can further omit the \( c_{0,0} \) term. Possible separatrices (dividing streamlines) of the critical point are given by \( \psi = 0 \), that is,
\[
\eta^2 + \frac{c_{2N,0}}{c_{0,2}} \xi^{2N} = 0.
\]  
(23)

There are two subcases: if \( c_{2N,0}/c_{0,2} > 0 \) the critical point is a degenerate centre, whereas if \( c_{2N,0}/c_{0,2} < 0 \) the critical point is a topological saddle (see Bakker, 1989).

**3.2. Unfolding of Degenerate Critical Points**

The flow patterns found in the previous section occur only when the parameters take certain values. The patterns are therefore structurally unstable. That is, a small change in the parameters will lead to qualitatively different pictures. To examine the possible bifurcations close to the simple linear degeneracy, we introduce a small parameter \( \varepsilon = c_{2,0} \).

Once again normal-form transformations will be used to simplify the system. As in the previous section nonlinear canonical transformations will be applied, but now transformations depend on the small parameter. We briefly show how to simplify fourth-order terms in the streamfunction, (7). A generating function is chosen,
\[
G = \xi y + s_{1,1,1} \xi \psi + s_{3,0,0} \xi^3 + s_{3,0,3} \eta^3 + s_{0,0,3} \xi^3 \psi + s_{3,1,0} \xi^3 \psi + s_{3,0,1} \xi y \psi + s_{0,3,1} \xi^3 \psi + s_{0,1,3} \xi^3 \psi + s_{1,3,0} \xi y^3,
\]
(24)
Theorem. Let \( c_2 \) and \( \bar{c}_{4,0}, \ldots, \bar{c}_{2N-2,0} \) be small coefficients. Assuming non-degeneracy conditions \( c_{0,2} \neq 0, \bar{c}_{2N,0} \neq 0 \) a normal-form of order \( 2N \) for the streamfunction (7) is

\[
\psi_{2N} = \frac{\sigma}{2} y^2 + h(x), \quad h(x) = c_{2,0} x^2 + \sum_{i=2}^{2N} \bar{c}_{2i,0} x^{2i},
\]

where \( \sigma = 2c_{0,2} \) and \( \bar{c}_{2i,0}, i = 2, \ldots, 2N \), are transformed coefficients.

Using the normal-form for \( \psi \), the corresponding differential equations

\[
\dot{x} = \sigma y, \quad \dot{y} = -h'(x),
\]

can be analysed. The origin is always a critical point and the total number of critical points is odd and at most \( 2N - 1 \). All critical points lie on the \( x \)-axis and are found as solutions to \( h'(x) = 0 \). At a critical point, the Jacobian is

\[
\begin{pmatrix}
0 & \sigma \\
-\bar{h}''(x) & 0
\end{pmatrix},
\]

and so the critical point is a centre if \( \sigma \bar{h}'''(x) > 0 \) and a saddle if \( \sigma \bar{h}''(x) < 0 \).

If the Jacobian is zero (i.e. \( \bar{h}''(x) = 0 \)), then the critical point is degenerate and bifurcation occurs. The type of critical point depends on \( \sigma \). The degenerate point is a centre if \( \sigma > 0 \) and a topological saddle if \( \sigma < 0 \). In addition to local bifurcations associated with degenerate critical points, there is also the possibility of global bifurcations such as heteroclinic trajectories which connect the critical points. Their conditions are illustrated in Figure 5.
3.3. Normal-Form of Order Four

For $2N = 4$, the streamfunction is

$$\psi_4 = \frac{\sigma}{2} y^2 + c_{2,0} x^2 + \sigma_{4,0} x^4. \quad (32)$$

$h'(x) = 2c_{2,0}x + 4\sigma_{4,0}x^3 = 0$ gives the critical points which are $x = 0$ and $x = \pm\sqrt{-c_{2,0}/2\sigma_{4,0}}$. For $c_{2,0}/2\sigma_{4,0} > 0$ and $c_{2,0} = 0$ the only critical point is the origin. For $c_{2,0}/2\sigma_{4,0} < 0$ there exist three critical points. Their types depend on $\sigma$.

For $\sigma > 0$ the origin is a centre for $c_{2,0}/2\sigma_{4,0} \geq 0$, while for $c_{2,0}/2\sigma_{4,0} < 0$ the origin is a saddle and the critical points $x = \pm\sqrt{c_{2,0}/2\sigma_{4,0}}$ are centres. For $\sigma < 0$ and $c_{2,0}/2\sigma_{4,0} \geq 0$ the origin is a saddle and for $c_{2,0}/2\sigma_{4,0} < 0$ the origin is a centre and the two critical points are saddles. A simple calculation shows that the streamfunction (32) has the same values at these saddles and hence there exist heteroclinic connections. The unfolding is shown in Figure 4.

Once again we examine values of coefficients in the expansion (7) for varying $A$ to determine the local flow patterns in the cavity. Indeed, as the aspect ratio $A$ is decreased from 0.9 a calculation shows:

(i) For $A \in (0.9, 0.318)$, $\sigma > 0$, $c_{2,0} > 0$ and $\sigma_{4,0} > 0$ and hence the origin is the only critical point and is a centre.

(ii) For $A \cong 0.318$, $\sigma > 0$ and $c_{2,0} = 0$. This shows that there is a bifurcation so the flow structure is unstable and the origin is a degenerate centre.

(iii) For $A \in (0.318, 0.169)$, $\sigma > 0$, $c_{2,0} < 0$ and $\sigma_{4,0} > 0$. There are now three critical points: the origin is a saddle point and other two $(x, y) = (\pm\sqrt{-c_{2,0}/2\sigma_{4,0}}, 0)$ are centres, Figure 4(c)(i).

The above information is summarized in Table 2. When $A \cong 0.169$, $c_{2,0}$ and $\sigma_{4,0}$ vanish and a new bifurcation occurs with two additional critical points. To determine the actual local flow patterns we take the normal-form of order six in streamfunction (29).

3.4. Normal-Form of Order Six

For $2N = 6$, the streamfunction is

$$\psi_6 = \frac{\sigma}{2} y^2 + c_{2,0} x^2 + \sigma_{4,0} x^4 + \sigma_{6,0} x^6, \quad (33)$$

where

$$\sigma_{6,0} = c_{6,0} + c_{4,0} - c_{2,0} \left( \frac{2c_{0,4}c_{4,0}}{5c_{0,2}} + \frac{c_{2,0}^2}{c_{0,2}} + \frac{c_{4,2}}{5c_{0,2}} + \frac{2c_{2,2}}{45c_{0,2}} \right)$$

$$+ c_{2,0}^2 \left( \frac{c_{0,4}c_{2,2}}{6c_{0,2}^2} + \frac{c_{2,4}}{5c_{0,2}^2} \right) + c_{2,0}^3 \left( \frac{13c_{0,4}^2}{20c_{0,2}^4} + \frac{c_{0,6}}{c_{0,2}^3} \right). \quad (34)$$

![Figure 4](image)

Figure 4. Local streamline patterns for $A = 0.5$ and $c_{2,0}/\sigma_{4,0} = 1.84 \times 10^{-2}$. 
By eliminating $x = 0$ and $x_{1,-2} = \pm \sqrt{\frac{-c_{4,0} \pm \sqrt{c_{4,0}^2 - 3c_{2,0}}}{3}}$, one obtains that bifurcations occur only at the origin when $c_{2,0} = 0$. In addition, for $A \in (0.169, 0.115)$ a calculation shows that at critical points $x_{1,-1} = \pm \sqrt{\frac{-c_{4,0} - \sqrt{c_{4,0}^2 - 3c_{2,0}}}{3}}$, $\sigma h(x_1) < 0$ and $\sigma h(x_{-1}) < 0$ (i.e. these are saddle points). The conditions of a global bifurcation, $h(x_1) = h(x_{-1})$ and $h'(x_1) = h'(x_{-1}) = 0$, are satisfied – hence there is a heteroclinic connection, see Figure 5(a). All relevant information is summarized in Table 3.

### 3.5. Normal-Form of Order Eight

For $2N = 8$, the streamfunction is

$$
\psi_8 = \frac{\sigma}{2} y^2 + c_{2,0} x^2 + \bar{c}_{4,0} x^4 + \bar{c}_{6,0} x^6 + \bar{c}_{8,0} x^8.
$$

By eliminating $x$ from $h'(x) = h''(x) = 0$ bifurcation conditions are given by

$$
48c_{2,0}\bar{c}_{8,0} = \left(3\bar{c}_{6,0} \pm \sqrt{16\bar{c}_{6,0}^2 - \frac{16}{3}\bar{c}_{4,0}}\right) \left(4\bar{c}_{6,0} \pm \frac{8}{3}\sqrt{16\bar{c}_{6,0}^2 - \frac{16}{3}\bar{c}_{4,0}}\right).
$$

---

**Table 2. Relevant information for the normal-form of a streamfunction of order four.**

<table>
<thead>
<tr>
<th>$A$</th>
<th>Coefficients</th>
<th>Critical points</th>
<th>Flow patterns</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A \in (0.9, 0.318)$</td>
<td>$\sigma &gt; 0, c_{2,0} &gt; 0, \bar{c}_{4,0} &gt; 0$</td>
<td>(0,0)</td>
<td>Centre</td>
</tr>
<tr>
<td>$A \equiv 0.318$</td>
<td>$\sigma &gt; 0, c_{2,0} = 0$</td>
<td>(0,0)</td>
<td>Degenerate centre</td>
</tr>
<tr>
<td>$A \in (0.318, 0.169)$</td>
<td>$\sigma &gt; 0, c_{2,0} &lt; 0, \bar{c}_{4,0} &gt; 0$</td>
<td>$(-\sqrt{-c_{2,0}/\bar{c}<em>{4,0}}, 0)$, $(\sqrt{-c</em>{2,0}/\bar{c}_{4,0}}, 0)$</td>
<td>Figure 4(c)(i)</td>
</tr>
</tbody>
</table>

---

**Figure 5.** Possible flow patterns for (a) five stagnation points and (b-c-d) seven stagnation points on $y = 0$. For each pattern the graph of $h(x)$ on $y = 0$ and its conditions are shown.
Table 3. Relevant information for the normal-form of a streamfunction of order six.

<table>
<thead>
<tr>
<th>A</th>
<th>Coefficients</th>
<th>Critical points</th>
<th>Flow patterns</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A \in (0.9, 0.318)$</td>
<td>$\sigma &gt; 0, c_{2,0} &gt; 0, \tau_{4,0} &gt; 0, \tau_{6,0} &gt; 0$</td>
<td>$(0, 0)$</td>
<td>Figure 4(a)(i)</td>
</tr>
<tr>
<td>$A \equiv 0.318$</td>
<td>$\sigma &gt; 0, c_{2,0} = 0$</td>
<td>$(0, 0)$</td>
<td>Degenerate centre</td>
</tr>
<tr>
<td>$A \in (0.318, 0.169)$</td>
<td>$\sigma &gt; 0, c_{2,0} &lt; 0, \tau_{4,0} &gt; 0, \tau_{6,0} &gt; 0$</td>
<td>$(x_{-1}, 0)$</td>
<td>Figure 4(c)(i)</td>
</tr>
<tr>
<td>$A \equiv 0.169$</td>
<td>$\sigma &gt; 0, c_{2,0} = \tau_{4,0} = 0$</td>
<td>$(0, 0)$</td>
<td>Degenerate centre</td>
</tr>
<tr>
<td>$A \in (0.169, 0.115)$</td>
<td>$\sigma &gt; 0, c_{2,0} &gt; 0, \tau_{4,0} &lt; 0, \tau_{6,0}^2 &gt; 3c_{2,0}$</td>
<td>$(x_{1.2}, 0)$</td>
<td>Figure 5(a)</td>
</tr>
</tbody>
</table>

Table 4. Relevant information for the normal-form of a streamfunction of order eight.

<table>
<thead>
<tr>
<th>A</th>
<th>Coefficients</th>
<th>Critical points</th>
<th>Flow patterns</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A \in (0.9, 0.318)$</td>
<td>$\sigma &gt; 0, c_{2,0} &gt; 0, \tau_{4,0} &gt; 0, \tau_{6,0} &gt; 0, \tau_{8,0} &gt; 0$</td>
<td>$(0, 0)$</td>
<td>Centre</td>
</tr>
<tr>
<td>$A \equiv 0.318$</td>
<td>$\sigma &gt; 0, c_{2,0} = 0$</td>
<td>$(0, 0)$</td>
<td>Degenerate centre</td>
</tr>
<tr>
<td>$A \in (0.318, 0.169)$</td>
<td>$\sigma &gt; 0, c_{2,0} &lt; 0, \tau_{4,0} &gt; 0, \tau_{6,0} &gt; 0$</td>
<td>$(x_{-1}, 0)$</td>
<td>Figure 4(c)(i)</td>
</tr>
<tr>
<td>$A \equiv 0.169$</td>
<td>$\sigma &gt; 0, c_{2,0} = \tau_{4,0} = 0$</td>
<td>$(0, 0)$</td>
<td>Degenerate centre</td>
</tr>
<tr>
<td>$A \in (0.169, 0.115)$</td>
<td>$\sigma &gt; 0, c_{2,0} &gt; 0, \tau_{4,0} &lt; 0, \tau_{6,0}^2 &gt; 3c_{2,0}$</td>
<td>$(x_{1.2}, 0)$</td>
<td>Figure 5(a)</td>
</tr>
<tr>
<td>$A \equiv 0.115$</td>
<td>$\sigma &gt; 0, c_{2,0} = \tau_{4,0} = \tau_{6,0} = 0$</td>
<td>$(0, 0)$</td>
<td>Degenerate centre</td>
</tr>
<tr>
<td>$A \in (0.115, 0.087)$</td>
<td>$\sigma &gt; 0, c_{2,0} &gt; 0, \tau_{4,0} &gt; 0, \tau_{6,0} &lt; 0, \tau_{8,0} &gt; 0$</td>
<td>$(x_{1.2.3}, 0)$</td>
<td>Figure 5(d)</td>
</tr>
</tbody>
</table>

For $A \in (0.9, 0.087)$ equation (37) is satisfied only when all coefficients vanish. This implies that bifurcations occur only at the origin. For $A \in (0.9, 0.115)$ we omit the details and the results are shown in Table 4. For $A \in (0.115, 0.087)$, the calculations show that $c_{2,0} < 0, \tau_{4,0} > 0, \tau_{6,0} < 0$ and $\tau_{8,0} > 0$ and hence seven critical points appear in the cavity. There are three alternative flow patterns as illustrated in Figure 5(b)–(d). The corresponding $h(x)$ and its conditions are given for each schematic. One might perhaps expect Figure 5(d) to represent the correct pattern since in only this case does the flow change continuously at the origin. Indeed, this is confirmed as follows. By eliminating $x_1, x_2$ from the conditions $h(x_1)h(x_2) < 0, h'(x_1)h'(x_2) = 0$ and $h''(x_1)h''(x_2) < 0$ one obtains

$$4c_{2,0}^2 + \left(4\tau_{6,0}^2 - \frac{1}{4}D\right)^2 \left(\tau_{6,0}^2 - \frac{1}{16}D\right) + 2c_{2,0}\left[-2\tau_{6,0} + \frac{1}{2}D\right]\left[\tau_{6,0} + \frac{1}{2}D\right] - \left[2\tau_{6,0} + \frac{1}{2}D\right]^2 \left(\tau_{6,0} - \frac{1}{4}D\right) > 0,$$

(38)

where $D = \sqrt{16\tau_{6,0}^2 - \frac{1}{2}\tau_{4,0}}$. Calculations show that (38) is satisfied for $A \in (0.115, 0.087)$. It is found that the second separatrix arises within the central section of the previous one.
4. Conclusion

Streamline patterns and their bifurcations for two-dimensional Stokes flow in a cavity having symmetric boundary conditions are investigated for a varying cavity aspect ratio $A$. Flow patterns are summarized for $A \geq 0.9$, and they confirmed the previous studies. For $A < 0.9$ new flow transformations are found using the analytical solution for the streamfunction expressed as an infinite series of Papkovich–Fadle eigenfunctions. However, as $A \to 0$ there are some difficulties in determining the actual flow structure. We used the normal-form transformations described by Brøns and Hartnack (1999) to simplify the streamfunction and then qualitative theory to determine the local flow pattern. It is found that bifurcations occur only at the origin and the flow structure consists of a series of nested separatrices, each contained within the central section of the previous one.

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References