LAPLACIAN SMOOTHING AND DELAUNAY TRIANGULATIONS

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SUMMARY
In contrast to most triangulation algorithms which implicitly assume that triangulation point locations are fixed, 'Laplacian' smoothing focuses on moving point locations to improve triangulation. Laplacian smoothing is attractive for its simplicity but it does require an existing triangulation. In this paper the effect of Laplacian smoothing on Delaunay triangulations is explored. It will become clear that constraining Laplacian smoothing to maintain a Delaunay triangulation measurably improves Laplacian smoothing.

An early reference to the use of Laplacian smoothing is to be found in a paper by Buell and Bush. They surveyed the use of Laplace's equation to generate two-dimensional meshes, and described an equipotential method which extends finite differences of Laplace's equation on rectangular grids to differences on triangular grids. On triangular grids, the weighted difference equations

$$\sum W_i(x_i-x) = 0, \quad \sum W_i(y_i-y) = 0$$

with weights $w_i$ are used, where $x_i$ and $y_i$ are the coordinates of the closest neighbours (connected by edges of triangles) to the coordinates of an internal vertex $(x,y)$. By applying the equipotential method with unit weights to arbitrary triangulations, the iterative solutions to the difference equations in equation (1) have become known as Laplacian smoothings of the triangulations.

Laplacian smoothing of triangular finite element meshes is commonly used in the following form. Let triangles $E_1, E_2, \ldots, E_k$ share vertex $z^*$, and let the remaining vertices of $E_1, \ldots, E_k$ be $z_1, \ldots, z_k$. Laplacian smoothing defines a new coordinate $z^*$ by the equation

$$z^* = (z_1 + \cdots + z_k)/k$$

The new coordinate for $z^*$ is immediately used for all subsequent Laplacian smoothing of other coordinates. Equation (2) is one step of an iterative solution to equation (1). Since $z^*$ will remain connected to $z_1, \ldots, z_k$ and equation (1) does not ensure that $z^*$ will remain within the triangulation domain, precautions are taken to guarantee that the new coordinate assigned to $z^*$ will define valid triangles.

Delaunay triangulations are useful finite element meshes because, of all triangulations of a random set of points, the Delaunay triangulation maximizes the smallest angle in its triangulation. This maximum minimum angle property of Delaunay triangulations is equivalent to a triangulation whose triangles have circumcircles which contain no other triangulation points in their interior. A variety of Delaunay triangulation algorithms are described in References 3-6; a very efficient algorithm is described in Reference 7. The algorithm in Reference 8 inserts new points at arbitrary locations and allows the location of already triangulated points to be changed, while maintaining the empty circumcircle property. This algorithm allows truly adaptive mesh generation, and this paper shows that its application measurably improves Laplacian smoothing.

Laplacian smoothing of Delaunay triangulations produces new triangulations which may no longer be Delaunay triangulations. Consequently, triangulation algorithms which require a Delaunay triangulation cannot accept additional triangulation points after Laplacian smoothing until a Delaunay triangulation is reconstructed. However, there is another, more serious deficiency of Laplacian smoothing: if an interior triangulation point $P$ is a vertex of $n$ triangles, $n>2$, then the maximum angle at $P$ must be greater than $360^\circ/n$, and no amount of Laplacian smoothing can
change this bound. This bound also implies that the maximum and minimum angles in triangulations commonly refined by connecting vertices of triangles to centroids of triangles cannot be less than $120^\circ$ and greater than $30^\circ$, respectively, even if Laplacian smoothing is used.

Laplacian smoothing remains an effective *ad hoc* technique for improving finite element meshes, especially meshes incorporating regions with different densities of elements. As an example, a graded mesh was created on a rectangle which was divided into cells, as shown in Figure 1(a). Points at the cell corners on the boundary were fixed, and within each cell a random point was produced by GGUBS, an IMSL subroutine which produces uniform distributions of random numbers. (See Reference 4 for an early example of using random number generators for finite element meshes.)

Figure 1. (a) The cellular decomposition. (b) The Delaunay triangulation of random numbers generated within each cell; undesirable Delaunay triangles are ringed. (c) The Laplacian smoothing of the Delaunay triangulation in (b). (d) The Laplace–Delaunay smoothing of the Delaunay triangulation in (b).
The Delaunay triangulation of these random points is shown in Figure 1(b), in which three obvious cases of undesirable triangles are ringed. The Laplacian-smoothed version of this triangulation is shown in Figure 1(c). Although Laplacian smoothing did not preserve the empty circumsphere property of the original triangles, most triangles in the Laplacian-smoothed triangulation are Delaunay triangles. Hence a Lawson type algorithm would be an efficient method of reconstructing the Delaunay triangulation.\(^{10, 11}\)

Each iteration of Laplacian smoothing can be improved by maintaining a Delaunay triangulation in the following way:

Process sequentially each interior triangulation point \(z\).

Step I: Use equation (1) to obtain new coordinates \(z^*\) for \(z\).

Step II: (a) If moving \(z\) to \(z^*\) preserves the Delaunay triangulation while maintaining the same connections to neighbouring triangulation points, move \(z\) to \(z^*\), update the data structures of the Delaunay triangulation and proceed to the next interior triangulation point.

(b) If moving \(z\) to \(z^*\) does not preserve the Delaunay retriangulation, with the same connections to neighbouring triangulation points, delete \(z\), locally retriangulate to maintain a Delaunay triangulation and insert \(z^*\) as a new triangulation point.

That a Delaunay triangulation is preserved in Step II (b) and that a local Delaunay retriangulation is sufficient is proved in Reference 6. Step II (b) not only maintains a Delaunay triangulation, but also allows new connections for interior triangulation points to be made. Figure 1(d) and Table I illustrate the potential for producing greater minimum and smaller maximum angles than is possible with Laplacian smoothing.

Since Laplacian smoothing tends to preserve Delaunay triangulations, the execution of Step II (b) was expected to be rare. However, Step II (b) was executed for approximately 75 per cent of the points in each iteration. The number of arithmetic operations then became a concern. The operation count for Steps I and II (a) are each \(O(k)\), where \(k\) is the number of vertices connected to \(z\). In Step II (b) the arithmetic operation count for the reinsertion of a single point can be \(O(n^2)\) unless appropriate data structures are used. Thus, the total operation count for Laplace-Delaunay smoothing can be \(O(n^3)\) per iteration, as opposed to \(O(n)\) for each iteration of Laplacian smoothing. However, the use of appropriate data structures can reduce the total operation count to \(O(Kn)\), where \(K\) is the maximum of number of triangles sharing any one vertex. Laplace-Delaunay smoothing does not appear to converge. Nevertheless, two iterations offered significant improvement over Laplace smoothing, a 45 per cent larger minimum angle and a 24 per cent smaller maximum angle. Extreme values of angles for additional iterations of Laplace-Delaunay smoothing fluctuated by approximately 1° from the values in the third iteration in Table I.

<table>
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<th>Iteration</th>
<th>Original triangulation</th>
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<th>Laplace-Delaunay smoothing</th>
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REFERENCES