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Some dependence results between the spreading speed and the coefficients of the space-time periodic Fisher-KPP equation

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We investigate in this paper the dependence relation between the space-time periodic coefficients A, q and μ of the reaction-diffusion equation

$$\partial_t u - \nabla \cdot (A(t,x)\nabla u) + q(t,x) \cdot \nabla u = \mu(t,x)u(1-u),$$

and the spreading speed of the solutions of the Cauchy problem associated with compactly supported initial data. We prove in particular that (1) taking the spatial or temporal average of μ decreases the minimal speed, (2) if μ is not constant with respect to x, then increasing the amplitude of the diffusion matrix A does not necessarily increase the minimal speed, (3) if $A = I_N$, μ is a constant, then the introduction of a space periodic drift term $q = \nabla Q$ decreases the minimal speed.

To prove these results, we use a variational characterization of the spreading speed that involves a family of periodic principal eigenvalues associated with the linearization of the equation near 0. We are thus back to the investigation of the dependence relation between this family of eigenvalues and the coefficients.

1 Introduction

1.1 General framework and definition of the spreading speed

This article investigates the asymptotic properties of the solutions of the space-time periodic Fisher-KPP equation:

$$\begin{cases} \partial_t u - \nabla \cdot (A(t,x)\nabla u) + q(t,x)\cdot \nabla u = \mu(t,x)u(1-u) \text{ in } \mathbb{R}^+ \times \mathbb{R}^N, \\ u(0,x) = u_0(x) \text{ in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $u_0 \not\equiv 0$ is a nonnegative, continuous and compactly supported initial datum. This equation arises in various models, that come from genetics, population dynamics, combustion or chemistry for examples. In these models, the function u represents the density of a population or of a chemical material. It diffuses in a space-time heterogeneous media through a diffusion matrix $A(t,x)$ and it reacts through a term $\mu(t,x)u(1-u)$, where $\mu(t,x)$ represents a growth rate at small density. Lastly, it is advected at a speed $q(t,x)$.

This equation has first been investigated in one-dimensional media, when A and μ do not depend on (t,x), $A > 0$, $\mu > 0$ and $q \equiv 0$, by Kolmogorov, Petrovski and Piskunov [21] and by Fisher [11] in the 30's, then in multidimensional media by Aronson and Weinberger [1] in the 70's. Among other properties, these authors proved that for all $e \in \mathbb{S}^{N-1}$, if u is the solution of (1.1) and $u_0 \not\equiv 0$ is a nonnegative, continuous and compactly supported function, then:

$$\begin{cases} \liminf\limits_{t\to+\infty} u(t,cte)=1 & \text{if } 0\leqslant c<2\sqrt{\mu A}, \\ \\ \lim\limits_{t\to+\infty} u(t,cte)=0 & \text{if } c>2\sqrt{\mu A}. \end{cases} \tag{1.2}$$

This result is called a *spreading property* and the speed $c^* = 2\sqrt{\mu a}$ is called a *spreading speed* in direction e.

In the late 70's, spreading properties have been proved for the space periodic Fisher-KPP equation by Freidlin and Gartner [13] and Freidlin [12]. In such media the spreading speed depends on the direction of propagation e and is defined in an implicit way. Namely, these authors proved that, if $\mu > 0$, $\nabla \cdot q = 0$, q has a null average and A, q and μ do not depend on t, then for all $e \in \mathbb{S}^{N-1}$, there exists a speed c^*_e such that

$$\begin{cases} \liminf\limits_{t\to+\infty} u(t,cte)=1 & \text{if } 0\leqslant c<c^*_e, \\ \\ \lim\limits_{t\to+\infty} u(t,cte)=0 & \text{if } c>c^*_e. \end{cases} \tag{1.3}$$

Moreover, they proved a useful variational formula for c^*_e that we will give in Section 1.3. The existence of pulsating traveling fronts[1] for the space periodic Fisher-KPP equation has been proved later under various hypotheses by Xin [32], Berestycki and Hamel [4] and Berestycki, Hamel and Roques [8]. These authors found a link between the spreading speed in direction e and the minimal speed of existence of all the pulsating traveling fronts of direction ξ with $e \cdot \xi > 0$.

The investigation of the dependence relations between the coefficients (A, q, μ) of the space periodic Fisher-KPP equation and the spreading speed $c^*_e = c^*_e(A, q, \mu)$ has started

[1] We refer to the references below for the definition of this notion.

at the beginning of the 2000's. There is a wide litterature on this topic, that we will describe in Section 2 below.

Recently, spreading properties in space-time periodic media have been proved when $N = 1$, $A = I_N$, $\mu > 0$ is a constant, $\nabla \cdot q = 0$ and q has a null average by Nolen, Rudd and Xin [25] and under the general hypotheses of Section 1.2 below by Weinberger [31] and by Berestycki, Hamel and Nadin [3]. The existence of pulsating traveling fronts in space-time periodic media has been proved by Nolen, Rudd and Xin [25] when $A = I_N$, q is incompressible and f does not depend on (t, x) and in a general framework by Nadin [23]. Hardly no dependence results between the spreading speed and the coefficients have been obtained in space-time periodic media.

In the present paper we give new dependence results between the spreading speed $c_e^* = c_e^*(A, q, \mu)$ and the space-time coefficients (A, q, μ). Some of these results are extension of results that are known from space periodic media to space-time periodic media. But most of the dependence relations that we prove here are new, even in space periodic media. Our main results are the following:

- taking the spatial or temporal average of μ decreases the minimal speed,
- if μ is not constant with respect to time, then increasing the amplitude of the diffusion matrix A does not necessarily increase the minimal speed,
- if $A = I_N$, μ is a constant, then the introduction of a space periodic drift term $q = \nabla Q$ increases the minimal speed.

Organization of the paper. In Section 1.2, we give the hypotheses we require on the coefficients (A, q, μ). Then, we define the family of periodic principal eigenvalues involved in the variational characterization of the spreading speed and we clearly state this characterization in Section 1.3. We state our results in Section 2. We also give a review of the known dependence relations between the coefficients (A, q, μ) and the spreading speed in this Section. Lastly, we prove our dependence relations with respect to μ in Section 3, with respect to A in Section 4 and with respect to q in Section 5.

1.2 Hypotheses

We assume that the diffusion matrix A, the advection term q and the growth rate μ are periodic in (t, x). That is, there exist some positive constant T and some vectors $L_1, ..., L_N$, where L_i is colinear to the axis of coordinates e_i, such that for all $i \in [1, N]$, for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$, one has:

$$\begin{array}{rclcl} A(t, x + L_i) & = & A(t + T, x) & = & A(t, x), \\ \mu(t, x + L_i) & = & \mu(t + T, x) & = & \mu(t, x), \\ q(t, x + L_i) & = & q(t + T, x) & = & q(t, x). \end{array}$$

We define the periodicity cell $C = \Pi_{i=1}^N (0, |L_i|)$. In the sequel the notion of periodicity will always refer to the periods $(T, L_1, ..., L_N)$.

We shall need some regularity assumptions on μ, A, q. The growth rate $\mu : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is supposed to be of class $\mathcal{C}^{\frac{\delta}{2}, \delta}$. The matrix field $A : \mathbb{R} \times \mathbb{R}^N \to S_N(\mathbb{R})$ is supposed to be of class $\mathcal{C}^{\frac{\delta}{2}, 1+\delta}$. We suppose futhermore that A is uniformly elliptic and continuous: there exist some positive constants γ and Γ such that for all $\xi \in \mathbb{R}^N$, $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ one has:

$$\gamma|\xi|^2 \leqslant \sum_{1\leqslant i,j\leqslant N} a_{i,j}(t,x)\xi_i\xi_j \leqslant \Gamma|\xi|^2, \tag{1.4}$$

where $|\xi|^2 = \xi_1^2 + ... + \xi_N^2$ and $a_{i,j}(t,x)$ is the coefficient (i,j) of the matrix $A(t,x)$. The drift term $q : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is supposed to be of class $\mathcal{C}^{\frac{\delta}{2},\delta}$ and we assume that $\nabla \cdot q \in L^\infty(\mathbb{R} \times \mathbb{R}^N)$. In the sequel, the direction of propagation $e \in \mathbb{S}^{N-1}$ will be fixed.

1.3 Characterization of the spreading speed by periodic principal eigenvalues

The characterization of the spreading speed involves the family of operators which is associated with exponentially decreasing solutions of the linearization of (1.1) in the neighborhood of 0:

$$L_\lambda\psi = \partial_t\psi - \nabla \cdot (A\nabla\psi) - 2\lambda A\nabla\psi + q \cdot \nabla\psi - (\lambda A\lambda + \nabla \cdot (A\lambda) + \mu - q \cdot \lambda)\psi, \tag{1.5}$$

where $\lambda \in \mathbb{R}^N$ and $\psi \in \mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^N)$. It has been proved (see [22] for example) that there exists a unique real number $k_\lambda(A,q,\mu)$ such that there exists a function $\psi \in \mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^N)$ that satisfies

$$\begin{cases} L_\lambda\psi = k_\lambda(A,q,\mu)\psi \text{ in } \mathbb{R}\times\mathbb{R}^N, \\ \psi > 0 \text{ in } \mathbb{R}\times\mathbb{R}^N, \\ \psi \text{ is periodic.} \end{cases} \tag{1.6}$$

We call $k_\lambda(A,q,\mu)$ the *space-time periodic principal eigenvalue* associated with operator L_λ.

The variational characterization of the spreading speed that we will use in the sequel has been proved by Berestycki, Hamel and the author [3] in two different ways:

Theorem 1.1 *[3] Assume that* $k_\lambda(A,q,\mu) < 0$ *for all* $\lambda \in \mathbb{R}^N$*. Then if* $u_0 \not\equiv 0$ *is a compactly supported, continuous and nonnegative initial datum and* u *is the associated solution of the Cauchy problem (1.1), one has*

$$\begin{cases} \liminf\limits_{t\to+\infty} u(t,cte) = 1 & \text{if } 0 \leqslant c < c_e^*(A,q,\mu), \\ \\ \lim\limits_{t\to+\infty} u(t,cte) = 0 & \text{if } c > c_e^*(A,q,\mu), \end{cases} \tag{1.7}$$

with

$$c_e^*(A,q,\mu) = \min_{\lambda\cdot e<0} \frac{k_\lambda(A,q,\mu)}{\lambda\cdot e}. \tag{1.8}$$

Remark. An immediate application of Theorem 3.7 of [22] yields that when $\nabla\cdot q = 0$, $\int_{(0,T)\times C} q = 0$ and $\int_{(0,T)\times C} \mu > 0$, then $k_\lambda(A,q,\mu) \leqslant -\int_{(0,T)\times C} \mu - \int_{(0,T)\times C} \lambda A \lambda < 0$ for all $\lambda \in \mathbb{R}^N$. Thus we can apply Theorem 1.1 in this case. In the sequel, this hypothesis will always be satisfied except in the second part of Section 2.4, where we will explain why $k_\lambda(A,q,\mu) < 0$ is still satisfied for all $\lambda \in \mathbb{R}^N$.

This formula highly simplifies the investigation of the dependence relation between (A,q,μ) and $c_e^*(A,q,\mu)$.

2 Statement of the dependence results

2.1 Spatial and temporal averaging of the growth rate

We begin with two comparison principles with the averaged media in x or in t:

Proposition 2.1 (Influence of the spatial variations) *If* A *and* q *do not depend on* x*, define*

$$\overline{\mu}(t) = \frac{1}{|C|}\int_C \mu(t,x)dx.$$

Then, if $\int_{(0,T)\times C}\mu \geqslant 0$*, the following comparison holds:*

$$c_e^*(A,q,\mu) \geqslant c_e^*(A,q,\overline{\mu}) = \min_{e\cdot\xi>0}\Big(\frac{2}{T}\sqrt{\int_0^T \xi A\xi \int_0^T\int_C \mu} - \frac{1}{T}\int_0^T q\cdot\xi.\Big) \tag{2.1}$$

Moreover, the equality holds if and only if μ *does not depend on* x*.*

The right-hand side of (2.1) only involves the space-time average $\int_{(0,T)\times C}\mu(t,x)dtdx$ of μ. Hence, taking the average in x erases the heterogeneity in time. Taking the temporal average of the growth rate also decreases the minimal speed of propagation but there is a qualitative difference since the dependence in x is not erased:

Theorem 2.2 (Influence of the temporal variations) *If* A *and* q *do not depend on* t*, define*

$$\hat{\mu}(x) = \frac{1}{T}\int_0^T \mu(t,x)dt.$$

Then, if $k_\lambda(A,q,\hat{\mu}) < 0$ *for all* $\lambda\in\mathbb{R}^N$*, the following comparison holds:*

$$c_e^*(A,q,\mu) \geqslant c_e^*(A,q,\hat{\mu}).$$

Moreover, the equality holds if and only if μ *can be written:* $\mu(t,x) = \mu_1(x) + \mu_2(t)$*.*

These results mean that, somehow, the heterogeneity in x or in t of the growth rate increases the speed of propagation (sse [7] for a discussion about the notion of heterogeneity). Using this heuristic definition of "heterogeneity", it is not true that heterogeneous drift or diffusion coefficient speed up the propagation. We will prove later that some compressible drifts may slow down the propagation. It has also been proved by Papanicolaou and Xin [28] that, in dimension 1, if b is a space periodic continuous function of average 0 and μ_0 is a positive constant, then $c^*_{e_1}(1+\delta b, 0, \mu_0) \leqslant c^*_{e_1}(1, 0, \mu_0) = 2\sqrt{\mu_0}$ when δ is small enough. Hence, taking the additive average of the diffusion or advection coefficients is not the good mean to quantify the heterogeneity of these terms. Other kind of averaging may give a positive result. For example, it has been proved by the author [24] that, for space periodic media, in dimension 1, $c^*_{e_1}(< a >_H, 0, \overline{\mu}) \leqslant c^*_{e_1}(a, 0, \mu)$, where $< a >_H$ is the harmonic average of a.

2.2 Influence of the amplitude of the reaction term

We first state that increasing the reaction term increases the speed of propagation. This is an easy extension of Proposition 1.15 of [6].

Proposition 2.3 *If $\mu_1 \geqslant \mu_2$, then for all A, q, one has:*

$$c^*_e(A, q, \mu_1) \geqslant c^*_e(A, q, \mu_2).$$

Moreover, the equality holds if and only if $\mu_1 \equiv \mu_2$.

Next, one can wonder what is the influence of the amplitude of the growth rate on the minimal speed.

Proposition 2.4 *1. Assume that* μ_0 *is a constant and consider a space-time periodic function* η*. If* $\int_{(0,T)\times C}\eta \geqslant 0$ *(resp.* $\int_{(0,T)\times C}\eta > 0$*), then* $B \mapsto c_e^*(I_N, 0, \mu_0 + B\eta)$ *is nondecreasing (resp. increasing). Moreover, if* $\int_{(0,T)\times C}\eta = 0$*, then* $B \mapsto c_e^*(I_N, 0, \mu_0 + B\eta)$ *is increasing if and only if* η *is not a constant with respect to* x*.*

2. Assume that A*,* q *and* μ *do not depend on* t *and that* $\max_{x\in\mathbb{R}^N}\int_0^T \eta(t,x)dt > 0$*. Then* $B \mapsto c_e^*(A, q, \mu + B\eta)$ *is increasing for* B *large enough.*

This result extends that of Berestycki, Hamel and Roques [8] from space periodic to space-time periodic media. In such media, the hypothesis of 2. only reads $\max_{\mathbb{R}^N}\eta > 0$. Hence, the main interest of Proposition 2.4 is to identify the generalization of this hypothesis to time-dependent media, that is, $\max_{x\in\mathbb{R}^N}\int_0^T \eta(t,x)dt > 0$.

2.3 Non-monotonicity with respect to the diffusion amplitude

It seems natural that increasing the diffusion coefficient may increase the speed of propagation. In [6], Berestycki, Hamel and Nadirashvili have proved that $\kappa \mapsto c_e^*(\kappa A, 0, \mu)$ is increasing if μ is constant and A only depends on x. It was an open problem to generalize this result to heterogeneous growth rate. El Smaily [9] has proved that such a result cannot hold if one includes an advection term q that depends on x. El Smaily has also proved that $A \geqslant B$ in the sense of positive matrix does not imply $c_e^*(A, 0, \mu) \geqslant c_e^*(B, 0, \mu)$ in general. But the monotonicity of the function $\kappa \mapsto c_e^*(\kappa A, 0, \mu)$ was still an open problem. We answer this question in the present paper by proving that such a monotonicity result does not hold in general.

Theorem 2.5 (Monotonicity with respect to the diffusion in space periodic media) *Assume that* A *and* μ *do not depend on* t*, that* μ *is not a constant, that* $q \equiv 0$ *and that* $\int_C \mu(x)dx \leqslant 0$*. Then, for all* $e \in \mathbb{S}^{N-1}$*, there exists* $M \in \mathbb{R}$ *(which depoends on* μ*), such that*

$$\kappa \mapsto c_e^*(\kappa A, 0, \mu + M) \text{ is not monotonic.}$$

Remark. In fact, we will prove that *for any* periodic function $\mu \in \mathcal{C}^{\delta}_{per}(\mathbb{R}^N)$ which does not depend on t and which is not a constant, there exists a sufficiently large M so that $\kappa \mapsto c_e^*(\kappa A, 0, \mu + M)$ is not monotonic, which is a stronger result.

Note that we do not ask A to be non-constant. Hence, this result means that $\kappa \mapsto c_e^*(\kappa I_N, 0, \mu)$ might be locally decreasing, which is a surprising fact. The heuristic explaination for this is that the population needs to stay during a sufficiently long time in the favourable zones, even if it means staying during a long time in the unfavourable ones, to get a high global reproduction rate.

When A only depends on x and μ only depends on t, we can extend the result of Berestycki, Hamel and Nadirashvili [6]:

Proposition 2.6 *Assume that* A *does not depend on* t*, that* $q \equiv 0$ *and that* μ *does not depend on* x*. Then, for all* $e \in \mathbb{S}^{N-1}$*,*

$$\kappa \mapsto c_e^*(\kappa A, 0, \mu) \text{ is increasing.}$$

In order to terminate this section, let us mention some dependence results for the function $\kappa \mapsto \frac{c_e^*(\kappa A, 0, \mu)}{\sqrt{\kappa}}$ when A and μ do not depend on t. The author has proved that this function is nonincreasing in [24]. He has also computed the limit of this function when $\kappa \to +\infty$. In dimension 1, the limit when $\kappa \to 0$ has been computed by Hamel, Roques and Fayard [14] when A and μ only take two values and by Hamel, Roques and the author [15] for general A, q and μ.

2.4 Influence of the drift

Incompressible drifts

It has been proved by Berestycki, Hamel and Nadirashvili [6] that, in space periodic media, the introduction of an incompressible drift with null average increases the propagation speed. Actually, the difficulty is to understand what it is the amplitude of this speed-up. It is known that this speed-up depends on the geometric properties of the level-lines of the flow associated with q (see [2, 5, 10, 16, 20, 30, 33, 34]).

We will not get into the details about general incompressible flows here and we only consider shear flows.

Proposition 2.7 *Assume that* μ_0 *is a positive constant and that the drift term can be written* $q(t,x) = (q_1(t,y), 0, ..., 0)$, *where one writes* $x = (x_1, y) \in \mathbb{R} \times \mathbb{R}^{N-1}$, $q_1 \not\equiv 0$ *and* $\int_C q_1 = 0$. *Then for all* $e \in \mathbb{S}^{N-1}$, $B \mapsto c_e^*(I_N, Bq, \mu_0)$ *is increasing.*

This monotonicity has been numerically observed by Nolen and Xin [26] in the case

$e = e_1$. There was no analytical proof of this numerical observation before, as far as we know. Moreover, the result is true in all direction e and not only when $e = e_1$.

Compressible drifts

There is hardly any paper on the influence of a compressible drift on the speed in the litterature. Only Nolen and Xin have investigated propagation problems in such media before [27]. For space stationary random drifts with mean zero, when $N = 1$, $A = 1$ and μ_0 is a positive constant. They have proved that:

$$\forall \alpha \in (0,1),\ \exists c,\ C_\alpha,\ \forall B > 0, \frac{c}{B} \leqslant c^*_{e_1}(1, Bq, \mu_0) \leqslant \frac{C_\alpha}{B^\alpha},$$

where $c^*_{e_1}(1, Bq, \mu_0)$ is the spreading speed in direction $e_1 = +1$ associated with the drift Bq.

We focus here on the drifts that can be written $q = \nabla Q$ in any dimension and prove that such drifts slow down the propagation.

Theorem 2.8 (Influence of a drift $q = \nabla Q$) *Assume that $A = I_N$, μ_0 is a positive constant, q does not depend on t, $\int_C q = 0$ and that q can be written $q = \nabla Q$. Then:*

1. $\dfrac{c^*_e(I_N, B\nabla Q, \mu_0)}{B} \to 0$ *as* $B \to +\infty$,
2. $c^*_e(I_N, B\nabla Q, \mu_0) \leqslant c^*_e(I_N, 0, \mu_0) = 2\sqrt{\mu_0}$.

Remarks. 1. In dimension 1, if $A = I_N$, the hypothesis is equivalent to $\int_C q = 0$. In dimension 2 or 3, if $A = I_N$, it is equivalent to $\int_C q = 0$ and *curl* $q = 0$.

2. We will prove that $k_\lambda(I_N, \nabla Q, \mu_0) = k_\lambda(I_N, 0, -\frac{1}{2}\Delta Q + \frac{1}{4}|\nabla Q|^2 + \mu_0)$. Hence,

Theorem 3.7 of [22] yields that $k_\lambda(I_N, \nabla Q, \mu_0) \leqslant -(\lambda^2+\mu_0)|C|T - \frac{1}{4}\int_{(0,T)\times C} |\nabla Q|^2 < 0$ for all $\lambda \in \mathbb{R}^N$ and thus the hypothesis of Theorem 1.1 is still satisfied.

3 Proof of the dependence results with respect to the growth rate

The aim of this section is to prove Proposition 2.1, Theorem 2.2, Proposition 2.3 and Proposition 2.4. We first give a direct proof of Proposition 2.3. Then we state some general dependence relations with respect to the growth rate that will enable us to prove the other results.

3.1 Proof of Proposition 2.3

Proof of Proposition 2.3. We use the same kind of proof as Berestycki, Hamel and Nadirashvili in [6]. If $\mu_1 \geqslant \mu_2$, one immediatly gets $k_\lambda(A,q,\mu_1) \leqslant k_\lambda(A,q,\mu_2)$ using the min-max characterization of $k_\lambda(A,q,\mu)$ (see [22] for example):

$$k_\lambda(A,q,\mu) = \max_{\phi\in\mathcal{C}^{1,2}_{per}(\mathbb{R}\times\mathbb{R}^N),\ \phi>0}\ \min_{\mathbb{R}\times\mathbb{R}^N}\Big(\frac{L_\lambda\phi}{\phi}\Big). \tag{3.1}$$

Thus $c^*_e(A,q,\mu_1) \geqslant c^*_e(A,q,\mu_2)$. Assume now that $c^*_e(A,q,\mu_1) = c^*_e(A,q,\mu_2)$ and take some $\lambda \in \mathbb{R}^N$ such that $\lambda\cdot e < 0$ and

$$c^*_e(A,q,\mu_1) = \frac{k_\lambda(A,q,\mu_1)}{\lambda\cdot e}.$$

One has:

$$c^*_e(A,q,\mu_1) = \frac{k_\lambda(A,q,\mu_1)}{\lambda\cdot e} = c^*_e(A,q,\mu_2) \leqslant \frac{k_\lambda(A,q,\mu_2)}{\lambda\cdot e},$$

and then $k_\lambda(A,q,\mu_1) \geqslant k_\lambda(A,q,\mu_2)$, that is, $k_\lambda(A,q,\mu_1) = k_\lambda(A,q,\mu_2)$.

Take now ϕ^1_λ some eigenfunction associated with μ_1 and ϕ^2_λ some eigenfunction associated with μ_2. Set $\kappa = \max_{(0,T)\times C} \dfrac{\phi_\lambda^1}{\phi_\lambda^2}$ and $z = \phi_\lambda^1 - \kappa\phi_\lambda^2$. This function is nonpositive, vanishes somewhere and satisfies:

$$\begin{aligned}&\partial_t z - \nabla\cdot(A\nabla z) - 2\lambda eA\nabla z + q\cdot\nabla z\\ &-(\lambda^2 eAe + \lambda\nabla\cdot(Ae) + \mu_1 - \lambda q\cdot e + k_{\lambda e}(A,q,\mu_1))z = \kappa(\mu_2-\mu_1)\phi_\lambda^2 \leqslant 0.\end{aligned}$$

Thus the periodicity in t and the strong parabolic maximum principle give $z \equiv 0$. Hence $\mu_1 \equiv \mu_2$.$\square$

3.2 Strict concavity of the principal eigenvalue

The concavity of $\mu \mapsto k_\lambda(A,q,\mu)$ has already been proved in [22], but we focus here on the strict concavity, which will be our main tool in order to investigate equality cases later.

Proposition 3.1 *For all A, q, μ_1, μ_2, $r \in (0,1)$ and $\lambda \in \mathbb{R}^N$, one has*

$$k_\lambda(A,q,r\mu_1+(1-r)\mu_2) \geqslant rk_\lambda(A,q,\mu_1) + (1-r)k_\lambda(A,q,\mu_2).$$

Moreover, if A and q do not depend on t, the equality holds if and only if $\mu_1 - \mu_2$ does not depend on x.

Proof. As we already mentionned it, the concavity has already been proved by the author in [22]. We include this proof here for the sake of completeness and because it will lead us to the strict concavity.

As we are considering any possible q, μ_1 and μ_2, we can assume[2] that $\lambda = 0$. Set $\mu = r\mu_1 + (1-r)\mu_2$ and consider ϕ_1 and ϕ_2 some periodic principal eigenfunctions associated with μ_1 and μ_2. Define $z_1 = \ln(\phi_1)$, $z_2 = \ln(\phi_2)$, $z = rz_1 + (1-r)z_2$ and $\phi = e^z$. One can compute:

$$\frac{\partial_t \phi - \nabla\cdot(A\nabla\phi) + q\cdot\nabla\phi}{\phi} = \partial_t z - \nabla\cdot(A\nabla z) - \nabla z A\nabla z + q\cdot\nabla z,$$

$$\begin{aligned}\text{and } \nabla z A \nabla z &= r\nabla z_1 A\nabla z_1 + (1-r)\nabla z_2 A\nabla z_2 - r(1-r)(\nabla z_1 - \nabla z_2)A(\nabla z_1 - \nabla z_2)\\ &\leqslant r\nabla z_1 A\nabla z_1 + (1-r)\nabla z_2 A \nabla z_2.\end{aligned} \tag{3.2}$$

Hence, for all $(t,x)\in\mathbb{R}\times\mathbb{R}^N$:

$$\begin{aligned}\frac{\partial_t \phi - \nabla\cdot(A\nabla\phi) + q\cdot\nabla\phi}{\phi} - \mu &\geqslant r(\partial_t z_1 - \nabla\cdot(A\nabla z_1) - \nabla z_1 A\nabla z_1 + q\cdot\nabla z_1 - \mu_1)\\ &\quad +(1-r)(\partial_t z_2 - \nabla\cdot(A\nabla z_2) - \nabla z_2 A\nabla z_2 + q\cdot\nabla z_2 - \mu_2)\\ &\geqslant r\Big(\frac{\partial_t\phi_1 - \nabla\cdot(A\nabla\phi_1) + q\cdot\nabla\phi_1}{\phi_1} - \mu_1\Big)\\ &\quad +(1-r)\Big(\frac{\partial_t\phi_2 - \nabla\cdot(A\nabla\phi_2) + q\cdot\nabla\phi_2}{\phi_2} - \mu_2\Big)\\ &\geqslant rk_0(A,q,\mu_1) + (1-r)k_0(A,q,\mu_2).\end{aligned}$$

Using the min-max characterization (3.1) of $k_0(A,q,\mu)$, we get

$$k_0(A,q,\mu) \geqslant rk_0(A,q,\mu_1) + (1-r)k_0(A,q,\mu_2). \tag{3.3}$$

This gives the first part of Proposition 3.1. Assume now that the equality holds. Then (3.2) is an equality for all $(t,x)\in\mathbb{R}\times\mathbb{R}^N$ and thus $\nabla z_1 \equiv \nabla z_2$. Write $z_1(t,x) =$

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[2] This is where the hypothesis "A and q do not depend on x" is used in the equality case.

$z_2(t,x)+f(t)$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}^N$. Then $\phi_1(t,x)=\phi_2(t,x)e^{f(t)}$ and

$$\begin{aligned}
0 &= \partial_t\phi_1-\nabla\cdot(A\nabla\phi_1)+q\cdot\nabla\phi_1-\mu_1\phi_1-k_0(A,q,\mu_1)\phi_1\\
&= e^{f(t)}\Big(f'(t)\phi_2+\partial_t\phi_2-\nabla\cdot(A\nabla\phi_2)+q\cdot\nabla\phi_2-\mu_1\phi_2-k_0(A,q,\mu_1)\phi_2\Big)\\
&= e^{f(t)}\Big(f'(t)\phi_2+(\mu_2-\mu_1)\phi_2+(k_0(A,q,\mu_2)-k_0(A,q,\mu_1))\phi_2\Big),
\end{aligned}$$

Hence:

$$\mu_2-\mu_1\equiv -f'(t)+k_0(A,q,\mu_1)-k_0(A,q,\mu_2),$$

and the right-hand side only depends on t.

On the other hand, if $\eta=\mu_1-\mu_2$ does not depend on x, for all $r\in(0,1)$, set

$$\psi(t,x)=\phi_2(t,x)\exp\Big(\int_0^t r\eta(s)ds-\frac{t}{T}\int_0^T r\eta(s)ds\Big).$$

This function is periodic in t and x and satisfies:

$$\partial_t\psi-\nabla\cdot(A\nabla\psi)+q\cdot\nabla\psi-(\mu_2+r\eta(t))\psi=(k_0(A,q,\mu_2)-\tfrac{r}{T}\int_0^T\eta(t)dt)\psi.$$

The uniqueness of the eigenelements gives:

$$k_0(A,q,r\mu_1+(1-r)\mu_2)=k_0(A,q,\mu_2+r\eta)=k_0(A,q,\mu_2)-\frac{r}{T}\int_0^T\eta(t)dt.$$

Thus for all $r\in(0,1)$:

$$k_0(A,q,r\mu_1+(1-r)\mu_2)=rk_0(A,q,\mu_1)+(1-r)k_0(A,q,\mu_2).$$

□

3.3 A general dependence result

In [8], Berestycki, Hamel and Roques proved that, if A,q and μ are constant, if η does not depend on t and if $\int_C\eta\geqslant 0$, then $B\mapsto k_\lambda(A,q,\mu+B\eta)$ is a nonincreasing function.

In order to prove some of our results, we need to extend this property to general heterogeneous coefficients. This extension involves the principal eigenfunction $\widetilde{\phi}_\lambda$ associated with the adjoint problem, defined up to multiplication by a positive constant by:

$$\begin{cases} -\partial_t\widetilde{\phi}_\lambda - \nabla\cdot(A\nabla\widetilde{\phi}_\lambda) + 2\lambda eA\nabla\widetilde{\phi}_\lambda - \nabla\cdot(q\widetilde{\phi}_\lambda) \\ -(\lambda\nabla\cdot(Ae) + \lambda^2 eAe - \lambda q\cdot e + \mu)\widetilde{\phi}_\lambda = k_\lambda(\mu)\widetilde{\phi}_\lambda, \\ \widetilde{\phi}_\lambda > 0, \\ \widetilde{\phi}_\lambda \text{ is periodic.} \end{cases} \tag{3.4}$$

We normalize this adjoint eigenfunction by $\int_{(0,T)\times C}\phi_\lambda\widetilde{\phi}_\lambda = 1$.

Proposition 3.2 *Take η a periodic continuous function. If $\int_{(0,T)\times C}\eta\phi_\lambda\widetilde{\phi}_\lambda \geqslant 0$ (resp. $\int_{(0,T)\times C}\eta\phi_\lambda\widetilde{\phi}_\lambda > 0$), then the function $B \mapsto k_\lambda(A,q,\mu+B\eta)$ is nonincreasing (resp. decreasing) over $\mathbb{R}^+$. Moreover, if $\int_{(0,T)\times C}\eta\phi_\lambda\widetilde{\phi}_\lambda = 0$ and if A, q and μ do not depend on t, then the function $B \mapsto k_\lambda(A,q,\mu+B\eta)$ is decreasing over $\mathbb{R}^+$ if and only if η is not a constant with respect to x.*

Proof. Set $F(B) = k_\lambda(A,q,\mu+B\eta)$. This function is concave and analytic from the Kato-Rellich theorem. It has been proved in Theorem 3.3 of [22] that

$$F'(0) = -\int_{(0,T)\times C}\eta\phi_\lambda\widetilde{\phi}_\lambda dtdx.$$

Thus if this quantity is negative, F is clearly decreasing over $\mathbb{R}^+$. If it is null, then F is nonincreasing.

Moreover, if A, q and μ do not depend on t, Proposition 3.1 yields that F is strictly concave if and only if η is not a constant with respect to x. Hence, if

$$F'(0) = -\int_{(0,T)\times C} \eta\phi_\lambda \widetilde{\phi}_\lambda dtdx = 0,$$

then F is decreasing if and only η is not a constant with respect to x. □

3.4 Applications of Proposition 3.2

We are now in position to prove our dependence results using Proposition 3.2.

Proof of Theorem 2.2. We set $\eta = \mu - \hat{\mu}$ and we apply Proposition 3.2, replacing μ by $\hat{\mu}$. The function $F : B \mapsto k_\lambda(A, q, \hat{\mu} + B\eta)$ is nonincreasing if $\int_{(0,T)\times C} \eta\phi_\lambda\widetilde{\phi}_\lambda \geqslant 0$, where ϕ_λ and $\widetilde{\phi}_\lambda$ are associated with the coefficients $(A, q, \hat{\mu})$. As $(A, q, \hat{\mu})$ do not depend on t, these eigenfunctions do not depend on t and thus:

$$\int_{(0,T)\times C} \eta\phi_\lambda\widetilde{\phi}_\lambda = \int_C (\int_0^T (\mu(t,x) - \hat{\mu}(x))dt)\phi_\lambda(x)\widetilde{\phi}_\lambda(x)dx = 0$$

since $\hat{\mu}(x) = \frac{1}{T}\int_0^T \mu(t,x)dt$. Thus:

$$F(1) = k_\lambda(A, q, \hat{\mu} + \eta) = k_\lambda(A, q, \mu) \leqslant F(0) = k_\lambda(A, q, \hat{\mu}).$$

As this is true for all $\lambda \in \mathbb{R}^+$, this gives:

$$c_e^*(A, q, \mu) \geqslant c_e^*(A, q, \hat{\mu}).$$

If the equality holds, considering $\lambda \in \mathbb{R}^N$ such that $\lambda \cdot e < 0$ and $c_e^*(A, q, \mu) = \frac{k_\lambda(A,q,\mu)}{\lambda\cdot e}$, one gets

$$\frac{k_\lambda(A, q, \mu)}{\lambda \cdot e} = c_e^*(A, q, \mu) = c_e^*(A, q, \hat{\mu}) \leqslant \frac{k_\lambda(A, q, \hat{\mu})}{\lambda \cdot e}.$$

Thus $k_\lambda(A, q, \hat{\mu} + \eta) = k_\lambda(A, q, \hat{\mu})$. Proposition 3.2 then gives that η does not depend on x. Thus $\mu(t,x) = \hat{\mu}(x) + \eta(t)$. □

Proof of Proposition 2.1. First of all, if A, q and μ do not depend on x, using Proposition 3.1 of [22], we have:

$$k_\lambda(A,q,\mu) = -\frac{1}{T}\int_0^T (\lambda A\lambda - \lambda\cdot q + \mu).$$

In order to compute $\min_{\lambda\cdot e<0} \frac{1}{\lambda\cdot e}\frac{1}{T}\int_0^T (\lambda A\lambda - \lambda\cdot q + \mu)$, we write $\lambda = \alpha\xi$, with $\xi \in \mathbb{S}^{N-1}$ and $\alpha < 0$. We compute:

$$\min_{\lambda\cdot e<0} \frac{1}{\lambda\cdot e}\frac{1}{T}\int_0^T (\lambda A\lambda - \lambda\cdot q + \mu)$$

$$= \min_{\xi\cdot e>0} \min_{\alpha<0} \big(\alpha\frac{1}{T}\int_0^T \xi A\xi - \frac{1}{T}\int_0^T \xi\cdot q + \frac{1}{\alpha}\frac{1}{T}\int_0^T \mu\big)$$

$$= \min_{e\cdot\xi>0} \frac{2}{T}\sqrt{\int_0^T \xi A\xi \int_0^T \mu} - \frac{1}{T}\int_0^T q\cdot\xi.$$

This gives the equality in (2.1).

Assume now that μ depends on x. Set $\eta = \mu - \overline{\mu}$. The same arguments as in the proof of Theorem 2.2 give $c_e^*(A,q,\mu) \geqslant c_e^*(A,q,\overline{\mu})$ and the equality holds if and only if η does not depend on x. In this case, $\mu = \overline{\mu} + \eta$ does not depend on x. □

Proof of Proposition 2.4. 1. This is an immediate consequence of Proposition 3.2 since, when $A = I_N$, $q \equiv 0$ and μ is a constant, one has $\phi_\lambda \equiv \widetilde{\phi}_\lambda \equiv 1$.

2. Set $\hat{\eta}(x) = \frac{1}{T}\int_0^T \eta(t,x)dt$. We know from Theorem 2.2 that

$$k_\lambda(A,q,\mu + B\eta) \leqslant k_\lambda(A,q,\mu + B\hat{\eta}).$$

Moreover, Berestycki, Hamel and Roques have proved in [8] that, as $\max_{x\in\mathbb{R}^N} \hat{\eta} > 0$, the right-hand side goes to $-\infty$ as $B \to +\infty$. Thus the left-hand side converges to $-\infty$ as

$B \to +\infty$. As it is a concave function of B, it is decreasing over $[B_0, \infty)$, with B_0 large enough. □

4 Proof of the non-monotonicity with respect to the diffusion term

In order to prove Theorem 2.5, we first show that $\kappa \mapsto k_0(\kappa A, 0, \mu)$ is increasing when μ is not a constant.

Lemma 4.1 *Assume that μ and A do not depend on t and that μ is not a constant. Then $\kappa \mapsto k_0(\kappa A, 0, \mu)$ is increasing.*

Proof. As $q \equiv 0$ and A and μ do not depend on t, the operator L_0 is self-adjoint. Hence, the Rayleigh characterization of the periodic principal eigenvalue reads

$$k_0(\kappa A, 0, \mu) = \min_{\phi \in \mathcal{C}^2_{per}(\mathbb{R}^N), \phi > 0} \frac{\int_C \Big(\kappa \nabla\phi A(x) \nabla\phi - \mu(x)\phi^2\Big) dx}{\int_C \phi^2 dx} \tag{4.1}$$

where $\mathcal{C}^2_{per}(\mathbb{R}^N)$ is the set of periodic functions of class $\mathcal{C}^2$. It follows from (4.1) that $\kappa \mapsto k_0(\kappa A, 0, \mu)$ is nondecreasing.

Assume that there exist $\kappa_1 > \kappa_2$ such that $k_0(\kappa_1 A, 0, \mu) = k_0(\kappa_2 A, 0, \mu)$. Let ϕ_1 the periodic principal eigenvalue associated with κ_1 and normalized by $\int_C \phi_1^2 = 1$. This function satisfies

$$-\kappa_1 \nabla \cdot (A(x)\nabla\phi_1) - \mu(x)\phi_1 = k_0(\kappa_1 A, 0, \mu)\phi_1. \tag{4.2}$$

Multiplying by ϕ_1 and integrating, we find that

$$\begin{array}{rcl} k_0(\kappa_1 A, 0, \mu) & = & \int_C \Big(\kappa_1 \nabla\phi_1 A(x)\nabla\phi_1 - \mu(x)\phi_1^2\Big) dx \\ & \geqslant & \int_C \Big(\kappa_2 \nabla\phi_1 A(x)\nabla\phi_1 - \mu(x)\phi_1^2\Big) dx \geqslant k_0(\kappa_2 A, 0, \mu) \end{array}$$

using (4.1). As $k_0(\kappa_1 A, 0, \mu) = k_0(\kappa_2 A, 0, \mu)$, all these inequalities are equalities and, in particular, $\nabla\phi_1 \equiv 0$. Hence, ϕ_1 is a constant and it follows from (4.2) that μ is a constant, which is a contradiction. Hence, $\kappa \mapsto k_0(\kappa A, 0, \mu)$ is increasing. □

Proof of Theorem 2.5. We first prove that for all μ, $k_\lambda(\kappa A, 0, \mu) \to -\max_{\mathbb{R}^N} \mu$ as $\kappa \to 0$. Using our min-max characterization (3.1), we get

$$k_\lambda(\kappa A, 0, \mu) \geqslant -\max_{\mathbb{R}^N} \mu - \kappa \|A\|_{W^{1,\infty}(\mathbb{R}\times\mathbb{R}^N)}(|\lambda| + |\lambda|^2).$$

Thus $\liminf_{\kappa\to 0} k_\lambda(A, 0, \mu) \geqslant -\max_{\mathbb{R}^N} \mu$ for all $\lambda \in \mathbb{R}^N$.

Next, it has been proved by the author in [24] that

$$k_{\rho e}(\kappa A, 0, \mu) = \min_{\alpha\in\mathcal{A}} \Big(\int_C \kappa \nabla\alpha A(x) \nabla\alpha - \int_C \mu(x)\alpha^2 - \rho^2 \kappa |C| D_e(\alpha^2 A) \Big),$$

where

$$\mathcal{A} = \{\alpha \in \mathcal{C}^1_{per}(\mathbb{R}^N),\ \alpha > 0,\ \int_C \alpha^2 = 1\}$$

and $D_e(A)$ is the effective diffusivity of a matrix field A in direction e, that is,

$$D_e(A) = \min_{\chi\in\mathcal{C}^1_{per}(\mathbb{R}^N)} \frac{1}{|C|} \int_C (e + \nabla\chi) A(x) (e + \nabla\chi). \tag{4.3}$$

Writing $\lambda = \rho e$ with $\rho > 0$ and $e \in \mathbb{S}^{N-1}$, we know that for all $\alpha \in \mathcal{A}$, one has

$$k_{\rho e}(\kappa A, 0, \mu) \leqslant \int_C \kappa \nabla\alpha A(x) \nabla\alpha - \int_C \mu(x)\alpha^2 - \rho^2 \kappa |C| D_e(\alpha^2 A).$$

This gives for all $\lambda \in \mathbb{R}^N$ and $\alpha \in \mathcal{A}$:

$$\limsup_{\kappa\to 0} k_\lambda(\kappa A, 0, \mu) \leqslant -\int_C \mu(x)\alpha^2.$$

Consider $x_0 \in \mathbb{R}^N$ such that $\max_{\mathbb{R}^N} \mu = \mu(x_0)$ and θ a smooth periodic test-function such that $\int_C \theta = 1$. Define $\alpha_n(x) = \sqrt{n^N \theta(n(x - x_0))}$. One has $\int_C \mu\alpha_n^2 \to \mu(x_0) = \max_{\mathbb{R}^N} \mu$ as $n \to +\infty$. But as $\alpha_n \in \mathcal{A}$ for all n, one has $\limsup_{\kappa\to 0} k_\lambda(\kappa A, 0, \mu) \leqslant -\int_C \mu(x)\alpha_n^2$ for all n and thus

$$\limsup_{\kappa\to 0} k_\lambda(\kappa A, 0, \mu) \leqslant -\max_{\mathbb{R}^N} \mu.$$

This gives $k_\lambda(\kappa A, 0, \mu) \to -\max_{\mathbb{R}^N} \mu$ as $\kappa \to 0$.

Lemma 4.1 yields that $\kappa \in (0,\infty) \mapsto k_0(\kappa A, 0, \mu)$ is an increasing function when μ is not a constant. Hence, for all $\kappa > 0$, there exists some small $\varepsilon > 0$ such that $k_0(\kappa A, 0, \mu) > -\max_{\mathbb{R}^N} \mu + 2\varepsilon$. Thus, for all $\lambda \in \mathbb{R}^N$ small enough, one has $k_\lambda(\kappa A, 0, \mu) > \varepsilon - \max_{\mathbb{R}^N} \mu$. Finally, this yields that $\kappa \mapsto k_\lambda(\kappa A, 0, \mu)$ is not nonincreasing when λ is small enough.

Fix such a small λ. Then, there exists $\kappa_1 > \kappa_2$ such that $k_\lambda(\kappa_1 A, 0, \mu) > k_\lambda(\kappa_2 A, 0, \mu)$. Using Proposition 3.1 in [24], we get that there exists $M \in \mathbb{R}$ such that $c_e^*(\kappa_1 A, 0, \mu + M) < c_e^*(\kappa_2 A, 0, \mu + M)$. This proves that $\kappa \mapsto c_e^*(\kappa A, 0, \mu + M)$ is not nondecreasing. □

Proof of Proposition 2.6. As A does not depend on t and μ does not depend on x, we immediatly get from the proof of Theorem 2.2 that

$$k_\lambda(\kappa A, 0, \mu) = k_\lambda(\kappa A, 0, 0) - \frac{1}{T}\int_0^T \mu(s)ds = \kappa k_\lambda(A, 0, 0) - \frac{1}{T}\int_0^T \mu(s)ds.$$

Moreover, we know that $k_\lambda(A, 0, 0) < k_0(A, 0, 0) = 0$ for all $\lambda \neq 0$ (see Proposition 3.2 of [22] together with the strict concavity with respect to λ). This concludes the proof. □

5 Proof of the dependence results with respect to the drift term

Proof of Proposition 2.7 The proof relies on the following observation, which holds for general diffusion matrix A, drift term q and reaction term μ that can be written $A = a(t,y)I_N$, $q(t,x) = (q_1(t,y), 0, ..., 0)$ and $\mu = \mu(t,y)$. In this case, for all direction of propagation $e = (e_1, \widetilde{e}) \in \mathbb{S}^{N-1}$, define $\widetilde{k}_\lambda(a, q_1, \mu)$ the periodic principal eigenvalue defined by the existence of a function $\varphi_\lambda \in \mathcal{C}^{1,2}_{per}(\mathbb{R}\times\mathbb{R}^{N-1})$ that solves

$$\begin{cases} \partial_t\varphi_\lambda - \nabla\cdot(a(t,y)\nabla\varphi_\lambda) - 2\lambda a(t,y)\widetilde{e}\cdot\nabla\varphi_\lambda \\ -(\lambda\nabla\cdot(a(t,y)\widetilde{e}) + \lambda^2 a(t,y) - \lambda q_1(t,y)e_1 + \mu(t,y))\varphi_\lambda = \widetilde{k}_\lambda(a,q_1,\mu)\varphi_\lambda \text{ in } \mathbb{R}\times\mathbb{R}^{N-1}, \\ \varphi_\lambda > 0 \text{ in } \mathbb{R}\times\mathbb{R}^{N-1}, \\ \varphi_\lambda \text{ is periodic in} t \text{ and } y. \end{cases} \tag{5.1}$$

Setting $\psi(t,x_1,y) = \varphi_\lambda(t,y)$, this function satisfies the eigenvalue problem (1.6) associated with L_λ. The uniqueness of the periodic principal eigenvalue yields $k_\lambda(aI_N,q,\mu) = \widetilde{k}_\lambda(a,q_1,\mu)$. Thus if $a\equiv 1$ and μ is a positive constant, we immediatly get from the proof of Proposition 2.4 that

$$B\mapsto k_\lambda(I_N,Bq,\mu_0) \text{ is decreasing,}$$

which concludes the proof. □

The proof of Theorem 2.8 uses a lemma of independent interest that we state separately:

Lemma 5.1 *For all coefficients (A,q,μ) that do not depend on t and $\lambda\in\mathbb{R}^N$, one has*

$$k_\lambda(A,q,\mu) \geqslant k_0(A,0,\frac{\nabla\cdot q}{2} + \lambda A\lambda - \lambda\cdot q + \mu).$$

Proof. We know that $L_\lambda\phi_\lambda = k_\lambda(A,q,\mu)\phi_\lambda$, where ϕ_λ does not depend on t. Multiplying this equation by ϕ_λ and integrating over C, this gives:

$$\begin{aligned} k_\lambda(A,q,\mu)\int_C \phi_\lambda^2 &= \int_C \nabla\phi_\lambda A\nabla\phi_\lambda + \tfrac{1}{2}\int_C (q-2A\lambda)\nabla(\phi_\lambda^2) \\ &\quad -\int_C (\lambda A\lambda + \nabla\cdot(A\lambda) - q\cdot\lambda + \mu)\phi_\lambda^2 \\ &= \int_C \nabla\phi_\lambda A\nabla\phi_\lambda - \int_C(\lambda A\lambda + \tfrac{\nabla\cdot q}{2} - q\cdot\lambda+\mu)\phi_\lambda^2. \end{aligned}$$

But we know from the Rayleigh characterization that if

$$X = \{\phi\in \mathcal{C}^2_{per}(\mathbb{R}^N),\ \phi>0,\ \int_C \phi^2 = 1\},$$

then:

$$\begin{aligned} & k_0(A,0,\tfrac{\nabla\cdot q}{2} + \lambda A\lambda - \lambda\cdot q + \mu) \\ & = \min_{\phi\in X}\int_C \nabla\phi A\nabla\phi - \int_{(0,T)\times C}(\lambda A\lambda + \tfrac{\nabla\cdot q}{2} - q\cdot\lambda + \mu)\phi^2. \end{aligned}$$

This gives the conclusion. □

Proof of Theorem 2.8. As ∇Q is periodic and $\int_C q = 0$, Q is periodic. Set ϕ_λ a positive eigenfunction associated with $k_\lambda(I_N,\nabla Q,\mu_0)$ and $\psi_\lambda(x) = \phi_\lambda(x)e^{-Q(x)/2}$. This new function satisfies:

$$\begin{cases} -\Delta\psi_\lambda - 2\lambda\cdot\nabla\psi_\lambda - (\mu_0 + |\lambda|^2 - \frac{1}{2}\Delta Q + \frac{1}{4}|\nabla Q|^2)\psi_\lambda = k_\lambda(I_N,\nabla Q,\mu_0)\psi_\lambda, \\ \psi_\lambda > 0, \\ \psi_\lambda \text{ is periodic.} \end{cases} \tag{5.2}$$

Using Lemma 5.1, we get:

$$\begin{aligned} k_\lambda(I_N,\nabla Q,\mu_0) &= k_\lambda(I_N,0,-\frac{1}{2}\Delta Q + \frac{1}{4}|\nabla Q|^2 + \mu_0) \\ &\geqslant k_0(I_N,0,|\lambda|^2 + \frac{1}{4}|\nabla Q|^2 - \frac{1}{2}\Delta Q + \mu_0) = -|\lambda|^2 - \mu_0, \end{aligned} \tag{5.3}$$

since $e^{-Q(x)/2}$ is the principal eigenfunction associated with this eigenvalue. This proves Theorem 2.8. □

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