Optimized Interface Conditions in Domain Decomposition Methods. Application to Layered Problems with Extreme Contrasts in the Coefficients

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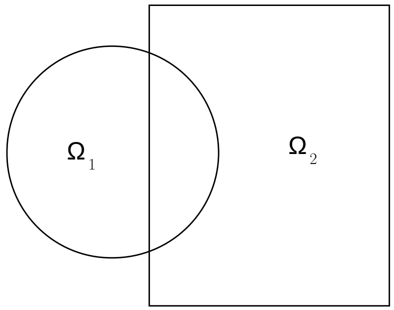
The First Domain Decomposition Method

The original Schwarz Method

(H.A. Schwarz, 1870)

$$-\Delta(u) = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega.$$



Multiplicative Schwarz Method : $(u_1^n, u_2^n) \rightarrow (u_1^{n+1}, u_2^{n+1})$ with

$$-\Delta(u_1^{n+1}) = f \quad \text{in } \Omega_1 \qquad\qquad -\Delta(u_2^{n+1}) = f \quad \text{in } \Omega_2$$
$$u_1^{n+1} = 0 \text{ on } \partial\Omega_1 \cap \partial\Omega \qquad\qquad u_2^{n+1} = 0 \text{ on } \partial\Omega_2 \cap \partial\Omega$$
$$u_1^{n+1} = u_2^n \quad \text{on } \partial\Omega_1 \cap \overline{\Omega_2}. \qquad\qquad u_2^{n+1} = u_1^{n+1} \quad \text{on } \partial\Omega_2 \cap \overline{\Omega_1}.$$

Easily parallelized algorithm, converges

but very slowly, overlapping subdomains only.

One possible Improvement: other interface conditions

(P.L. Lions, 1988)

$$
\begin{aligned}
&-\Delta(u_1^{n+1}) = f \quad \text{in } \Omega_1, \\
&u_1^{n+1} = 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega, \\
&(\frac{\partial}{\partial n_1} + \alpha)(u_1^{n+1}) = (-\frac{\partial}{\partial n_2} + \alpha)(u_2^n) \quad \text{on } \partial\Omega_1 \cap \overline{\Omega_2},
\end{aligned}
$$

(n_1 and n_2 are the outward normals to the subdomains' boundaries)

$$
\begin{aligned}
&-\Delta(u_2^{n+1}) = f \quad \text{in } \Omega_2, \\
&u_2^{n+1} = 0 \quad \text{on } \partial\Omega_2 \cap \partial\Omega \\
&(\frac{\partial}{\partial n_2} + \alpha)(u_2^{n+1}) = (-\frac{\partial}{\partial n_1} + \alpha)(u_1^n) \quad \text{on } \partial\Omega_2 \cap \overline{\Omega_1}.
\end{aligned}
$$

with $\alpha > 0$. Overlap is not necessary for convergence.

Extended to the Helmholtz equation (B. Després, 1991; Collino, Ghanemi and Joly, 2000)

a.k.a Two-Lagrange Multiplier FETI Method, 1998.

Convergence with interface conditions involving second order tangential derivatives :

$$\left(\frac{\partial}{\partial n_i} + \alpha - \frac{\partial}{\partial \tau}\gamma\frac{\partial}{\partial \tau}\right)$$

The proof was made for a finite volume discretization of the problem. We give here the continuous version of the result.

Let us consider the following problem

$$\eta(\mathbf{x})u - div(\kappa(\mathbf{x})\nabla u) = f \quad \text{ in } \Omega,$$
$$u = 0 \quad \text{ on } \partial\Omega,$$

with $\eta(\mathbf{x}), \kappa(\mathbf{x}) > C > 0$.
The domain is decomposed into N nonoverlapping subdomains $(\Omega_i)_{1 \leq i \leq N}$.
Let Γ_{ij} be the interface $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$.

Convergence

The interface condition reads

$$\kappa(\mathbf{x})\frac{\partial u_i^{n+1}}{\partial n_i} + \alpha_{ij}(\mathbf{x})u_i^{n+1} - \frac{\partial}{\partial \tau_i}(\gamma_{ij}(\mathbf{x})\frac{\partial u_i^{n+1}}{\partial \tau_i})$$
$$= -\kappa(\mathbf{x})\frac{\partial u_j^{n}}{\partial n_j} + \alpha_{ij}(\mathbf{x})u_j^{n} - \frac{\partial}{\partial \tau_j}(\gamma_{ij}(\mathbf{x})\frac{\partial u_j^{n}}{\partial \tau_j}) \text{ on } \Gamma_{ij}.$$

with

$$\alpha_{ij}(\mathbf{x}) = \alpha_{ji}(\mathbf{x}) \geq \alpha_0 > 0,$$

$$\gamma(\mathbf{x})_{ij} = \gamma(\mathbf{x})_{ji} \geq 0 \text{ and } \gamma_{ij}(\mathbf{x}) = 0 \text{ on } \partial\Gamma_{ij}$$

Let us denote

$$\Lambda_{ij} = \alpha_{ij}(\mathbf{x}) - \frac{\partial}{\partial \tau_i}(\gamma_{ij}(\mathbf{x})\frac{\partial}{\partial \tau_i}), \quad \mathbf{x} \in \Gamma_{ij}.$$

Lemma 1 *The algorithm converges in the H^1 norm (discrete).*

The convergence rate depends on α et γ, how to choose them?

Plan

1. Optimal Interface Conditions
2. Optimized Interface Conditions
3. Application to the Helmholtz equation
4. Optimized IC Discontinuous coefficients equations
5. Conclusion

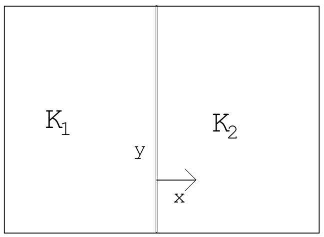
Optimal Interface Conditions for Heterogeneous Domain Decomposition

Let us consider the problem

$$\begin{aligned}
&\mathcal{L}_i(P_i) = f \quad \text{in } \Omega_i, \quad i = 1,2 \\
&P_1 = P_2 \quad \text{on } \Gamma_{12}, \\
&\kappa_1 \frac{\partial P_1}{\partial n_1} + \kappa_2 \frac{\partial P_2}{\partial n_2} = 0 \quad \text{on } \Gamma_{12}.
\end{aligned}$$

where

$$\mathcal{L}_i = \eta_i - div(\kappa_i \nabla \dot{)}$$



Optimal Interface Conditions for Heterogeneous Domain Decomposition

Let

$$u_i = \kappa_i \nabla P_i$$

Let us consider a Schwarz method:

$$\mathcal{L}_1(P_1^{n+1}) = f \quad \text{in } \Omega_1 \qquad\qquad \mathcal{L}_2(P_2^{n+1}) = f \quad \text{in } \Omega_2$$

$$P_1^{n+1} = 0 \text{ on } \partial\Omega_1 \cap \partial\Omega \qquad\qquad P_2^{n+1} = 0 \text{ on } \partial\Omega_2 \cap \partial\Omega$$

$$u_1^{n+1}.\vec{n}_1 + \mathcal{B}_1(P_1^{n+1}) \qquad\qquad u_2^{n+1}.\vec{n}_2 + \mathcal{B}_2(P_2^{n+1})$$

$$= -u_2^n.\vec{n}_2 + \mathcal{B}_1(P_2^n) \quad \text{on } \Gamma_1 \qquad\qquad = -u_1^n.\vec{n}_1 + \mathcal{B}_2(P_1^n) \quad \text{on } \Gamma_2$$

We take

$$\mathcal{B}_1 = DtN_2.$$

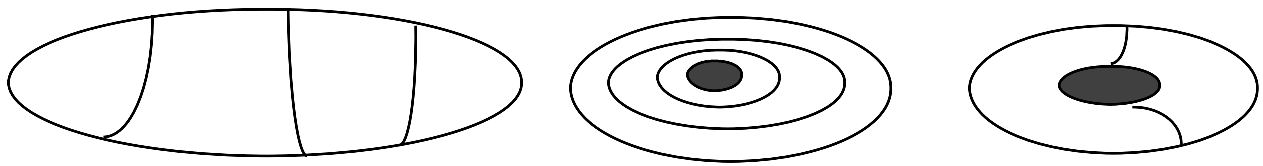
and have convergence in two iterations.

Optimal Interface Conditions for Heterogeneous Domain Decomposition

(Rogier, de Sturler and N., 1993)

The result can be generalized to variable coefficients operators and a decomposition of the domain Ω in more than two subdomains.

For the following geometries,



one can define interface conditions such as to have convergence in a number of iterations equals to the number of subdomains.

For arbitrary decompositions, negative conjectures have been formulated (F. Nier, *Séminaire X-EDP*, 1998).

Optimized Interface Conditions in the scalar case

The Steklov-Poincaré map DtN is not a partial differential operator. It is

1. non local
2. its explicit form is unknown in the general case

It is approximated by a partial differential operator

$$DtN \simeq \alpha_{opt} - \frac{\partial}{\partial \tau}(\gamma_{opt} \frac{\partial}{\partial \tau})$$

The parameters are chosen in order to minimize the convergence rate of the algorithm. (Tan, Mart, Borsboom, 1994)
These interface conditions are called optimized of order 2 conditions (opt2)
If we take $\gamma = 0$, the optimization is performed only w.r.t. α, they are called optimized of order 0 (opt0)

Application: the Helmholtz Equation

Joint work with M. Gander and F. Magoulès, (2002)

$$(-\omega^2 - \Delta)(u) = f$$

By choosing carefully the coefficients α and γ, it is possible to optimize the convergence rate of the iterative method which in the Fourier space is given by

$$\rho(k;\alpha,\gamma) \equiv \begin{cases} \left| \dfrac{I\sqrt{\omega^2 - k^2} - (\alpha + \gamma k^2)}{I\sqrt{\omega^2 - k^2} + (\alpha + \gamma k^2)} \right| & \text{if } |k| < \omega \quad (I^2 = -1) \\ \\ \left| \dfrac{\sqrt{k^2 - \omega^2} - (\alpha + \gamma k^2)}{\sqrt{k^2 - \omega^2} + (\alpha + \gamma k^2)} \right| & \text{if } \omega < |k| < \pi/h \end{cases}$$

Finally, we get analytic formulas for α and γ :

$$\alpha_{opt} = \alpha(\omega, h) \text{ and } \gamma_{opt} = \gamma(\omega, h),$$

where h is the mesh size.

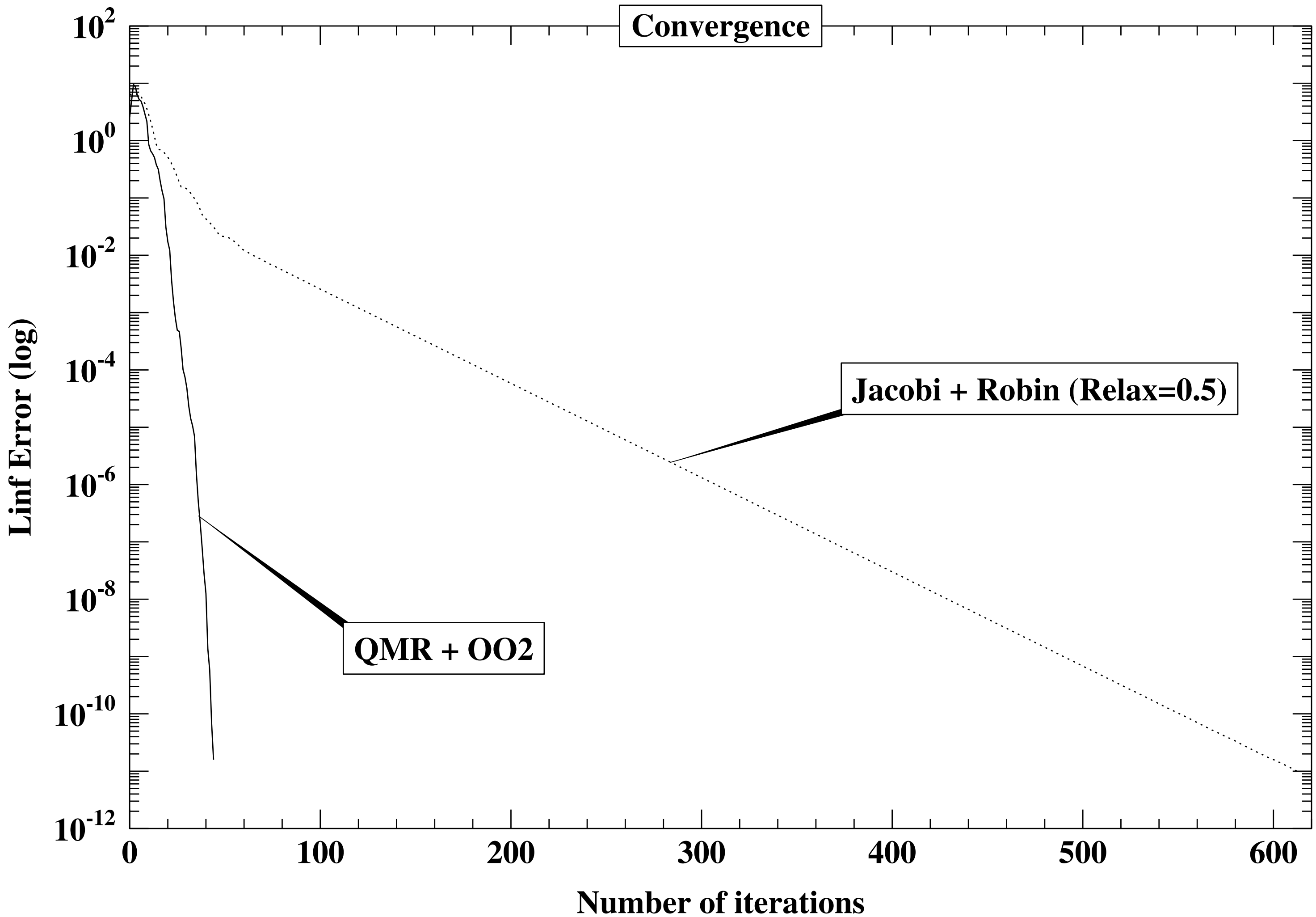
A Krylov method replaces the fixed point algorithm

Schwarz method
with arbitrary interface conditions
$\mathcal{C}_1$ and $\mathcal{C}_2$

$$
\begin{aligned}
&\mathcal{L}(u_1^{n+1}) = f \quad \text{in } \Omega_1,\\
&u_1^{n+1} = 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega,\\
&\mathcal{C}_1(u_1^{n+1}) = \mathcal{C}_1(u_2^n) \quad \text{on } \partial\Omega_1 \cap \overline{\Omega_2},\\
&\mathcal{L}(u_2^{n+1}) = f \quad \text{in } \Omega_2,\\
&u_2^{n+1} = 0 \quad \text{on } \partial\Omega_2 \cap \partial\Omega,\\
&\mathcal{C}_2(u_2^{n+1}) = \mathcal{C}_2(u_1^n) \quad \text{on } \partial\Omega_2 \cap \overline{\Omega_1},
\end{aligned}
\quad \Longleftrightarrow \quad
\begin{array}{c}
Sub(\mathcal{C})(\lambda) = b \text{ solved by Jacobi}\\
\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \lambda_i = \mathcal{C}_i(u_j),\ i \neq j\\
\Downarrow\\
\text{CG, GMRES, BICG, } \ldots
\end{array}
$$

The Helmholtz Equation – Numerical Results

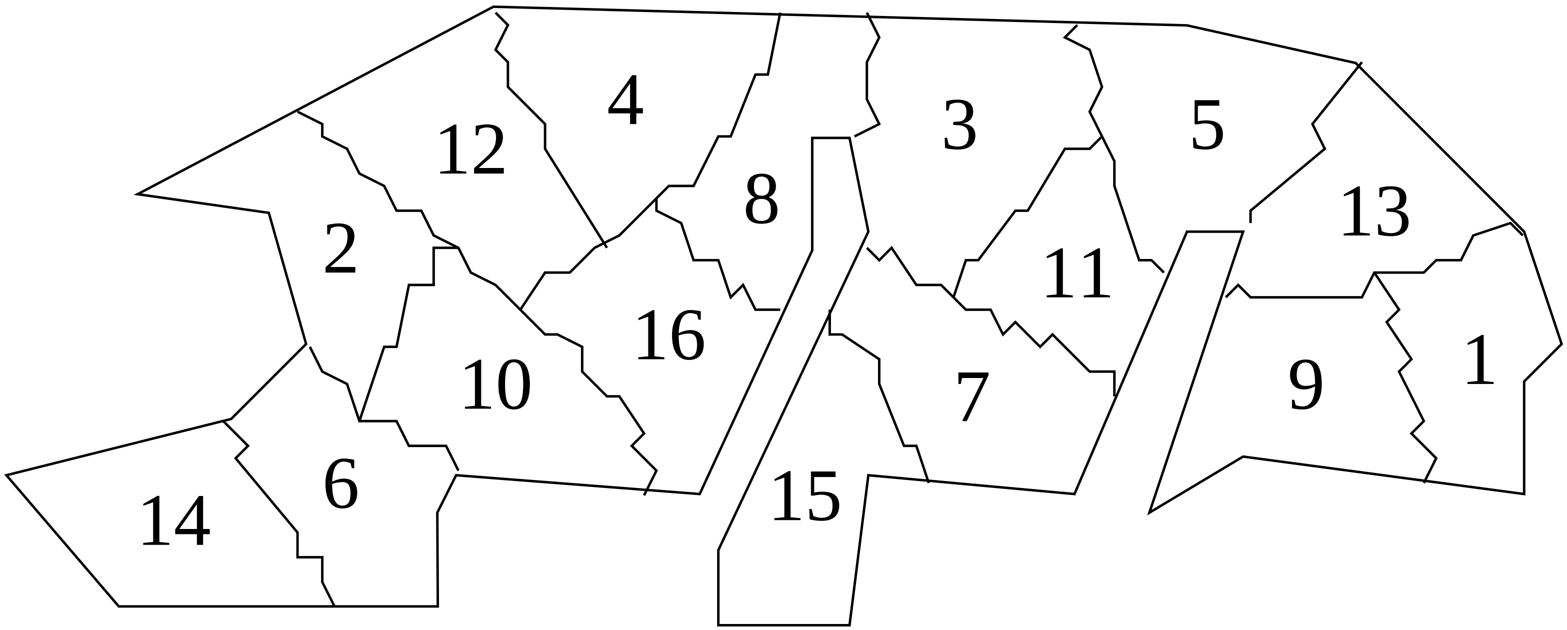
Waveguide: Optimized Schwarz method with QMR compared to ABC0 ($\partial_n + I\omega$) with relaxation on the interface



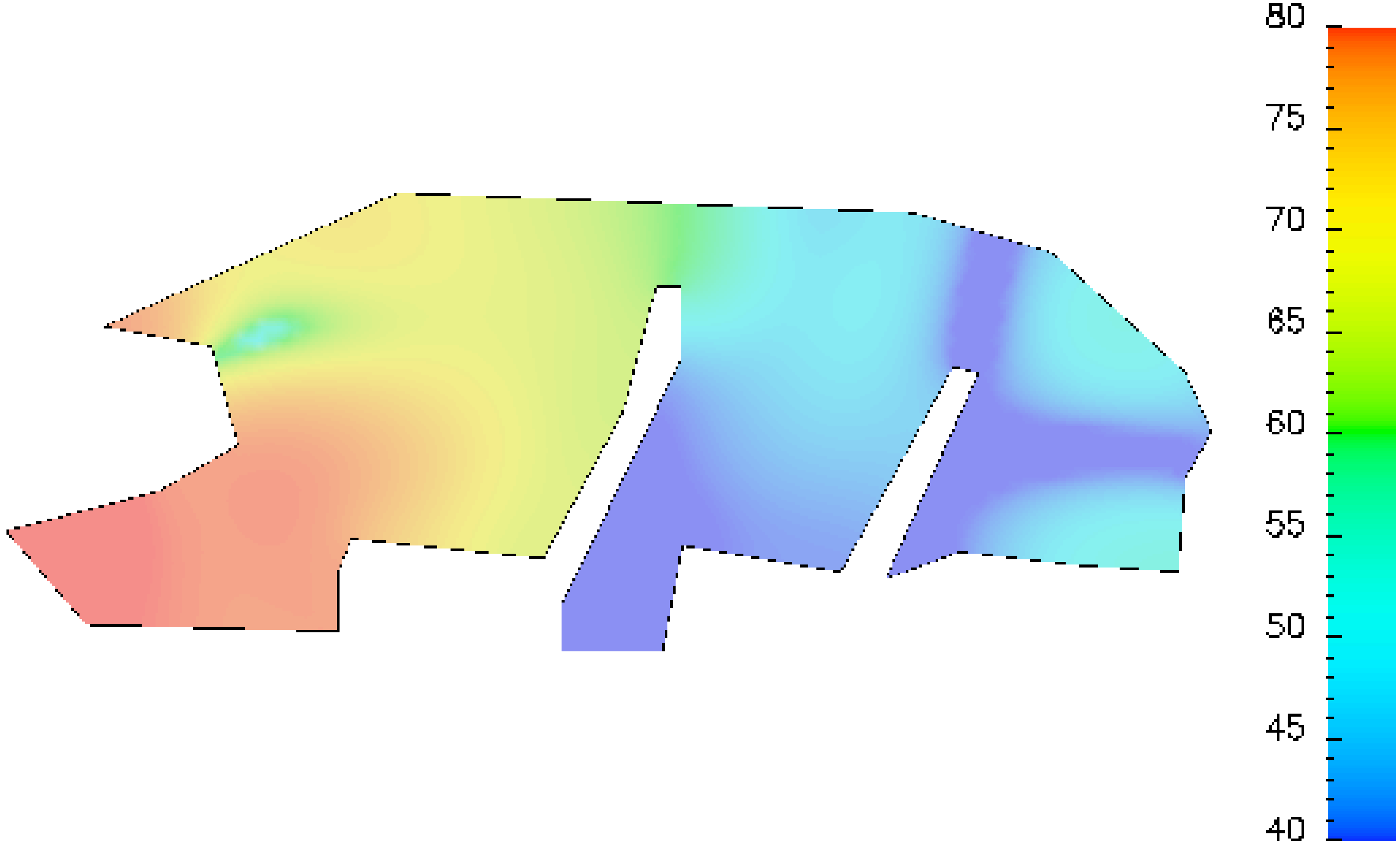
Chevalier and N., 1997

Application: the Helmholtz Equation

Numerical Results: Acoustic in a Car



Numerical Results: Acoustic in a Car



The Helmholtz Equation – Numerical Results

Acoustic in a Car : Iteration Counts for various interface conditions

N_s	ABC 0	ABC 2	Optimized
2	16 it	16 it	9 it
4	50 it	52 it	15 it
8	83 it	93 it	25 it
16	105 it	133 it	34 it

ABC 0 Absorbing Boundary Conditions of Order 0 $(\partial_n + I\omega)$ (Desprès, 1991, Casarin, Elliot and Widlund, 1997, McInnes, Susan-Resigna, Keyes Atassi, 1998, Farhat, Macedo, Lesoinne, 2000)

ABC 2 Absorbing Boundary Conditions of Order 2 $(\partial_n + I\omega - 1/(2I\omega)\partial_{y^2})$, (Meade, 1997)

Optimized Optimized Interface Conditions

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1. Optimal Interface Conditions
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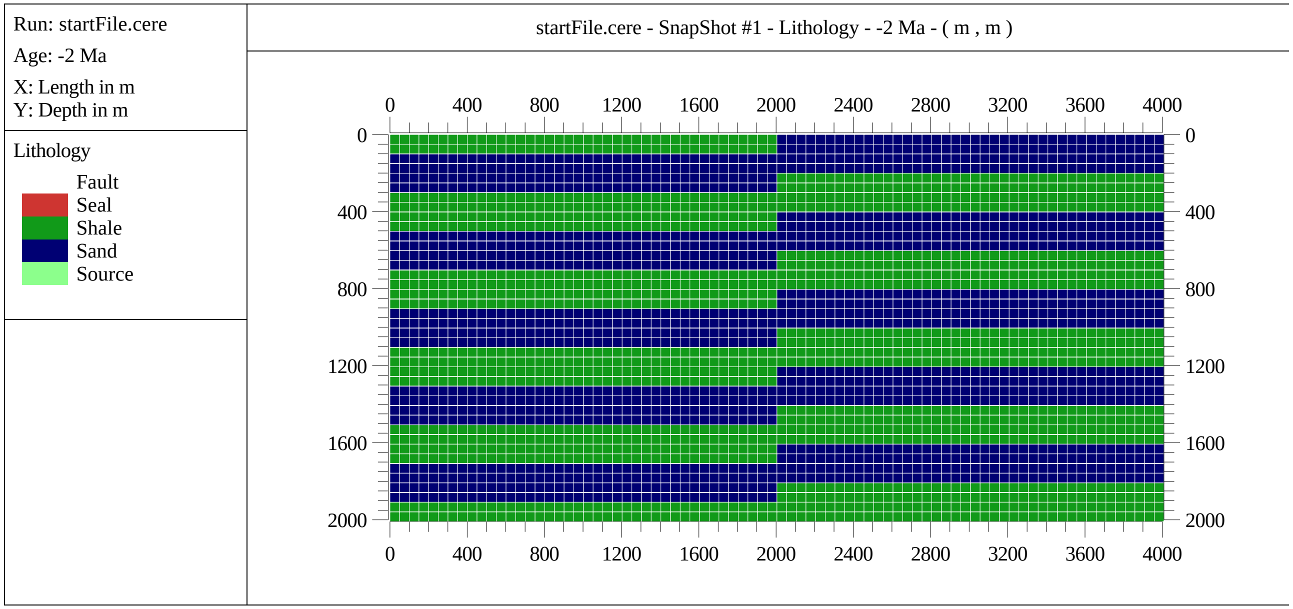
4. Optimized IC Discontinuous coefficients equations
    - The equation
    - "Frozen" coefficients interface conditions
    - Deflation or coarse grid
    - Generalized Neumann-Neumann/ FETI algorithm
    - Optimized of Order 2 Interface Conditions
5. Conclusion

Highly Heterogeneous media

Joint work with Eric Flauraud (IFP)

$$\eta u - div(\kappa \nabla u) = f$$



Jumps in the coefficients η et κ of four orders of magnitude.

Anisotropic tensor κ: $10^{-4} \leq \kappa_x/\kappa_y \leq 10^4$.

Discontinuous coefficients both across and along the interface.

$\Rightarrow$ Plateau in the convergence of Krylov methods even with “good” preconditioners.

ICs for discontinuous coefficients at the continuous level

$$\mathcal{L}_i(P) = \eta_i P - \frac{\partial}{\partial x}(C_i \frac{\partial}{\partial x} P) - \frac{\partial}{\partial y}(\kappa_{i,y} \frac{\partial}{\partial y} P), \qquad i = 1, 2$$

with η_i, C_i and $\kappa_{i,y}$ independant of the x variable.
Let $\Omega_1 = \mathbb{R}^- \times]0, L[$, $\Omega_2 = \mathbb{R}^+ \times]0, L[$ and $\Gamma = \{0\} \times]0, L[$. Consider a Schwarz type method:

$\mathcal{L}_1(P_1^{n+1}) = f \quad$ in Ω_1

$P_1^{n+1} = 0$ on $\partial\Omega_1 \backslash \Gamma$

$(C_1 \partial_x + \Lambda_{ap,2})(P_1^{n+1})$

$= (-C_2 \partial_x + \Lambda_{ap,2})(P_2^n) \quad$ on Γ

$\mathcal{L}_2(P_2^{n+1}) = f \quad$ in Ω_2

$P_2^{n+1} = 0$ on $\partial\Omega_2 \backslash \Gamma$

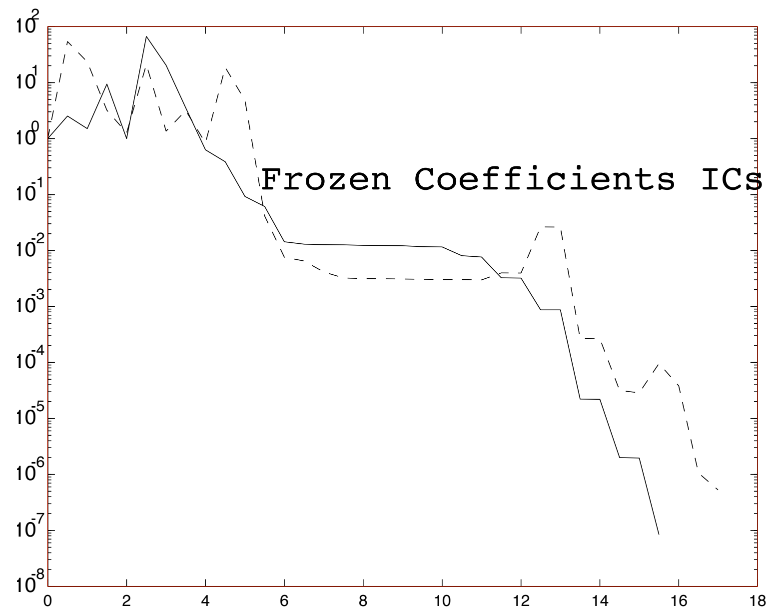
$(C_2 \partial_x + \Lambda_{ap,1})(P_2^{n+1})$

$= (-C_1 \partial_x + \Lambda_{ap,1})(P_1^n) \quad$ on Γ

We seek a partial differential approximation to the DtN_i map:

$$\Lambda_{ap,i} \simeq C_i^{1/2} \sqrt{C_i^{-1/2}(\eta_i - \frac{\partial}{\partial y}(\kappa_{i,y} \frac{\partial}{\partial y}))C_i^{-1/2}} C_i^{1/2} \ (:= \Lambda_i)$$

“Frozen” coefficients ICs



residual vs. iteration counts for various interface conditions (IC)

- Willien, Faille, Nataf, Schneider. Domain decomposition methods for fluid flow in porous medium.(1998)
- Roux, Magoulès, Series, Boubendir. Approximation of optimal interface b.c. for two-lagrange multiplier feti method. (2004)
- Genseberger, Sleijpen, van der Vorst. Using domain decomposition in the Jacobi-Davidson method (2002)

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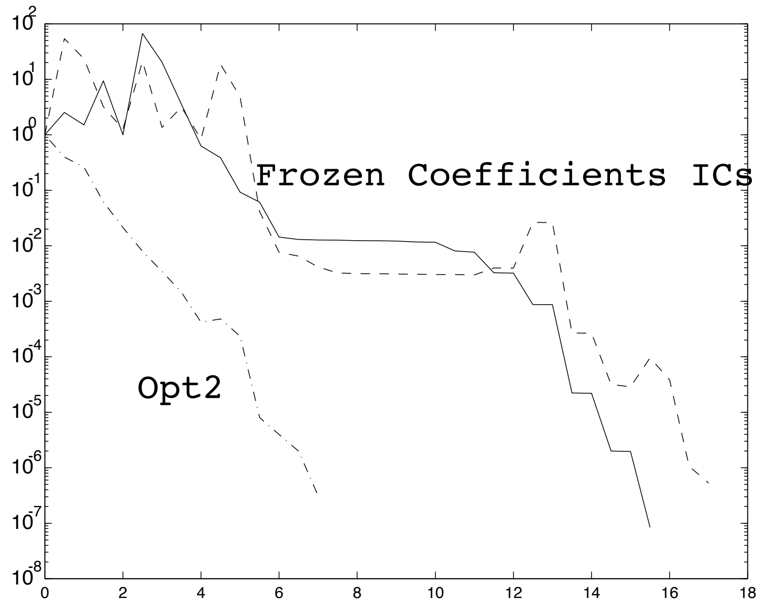

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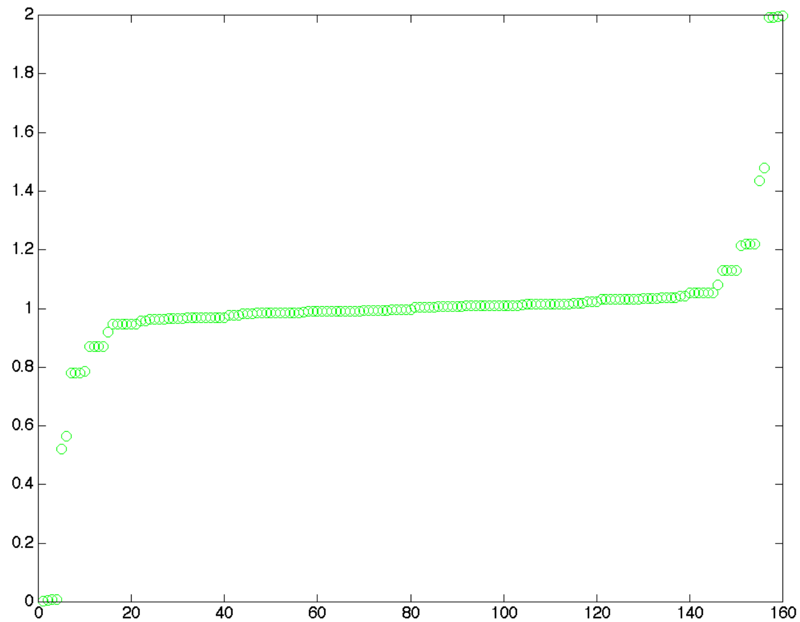
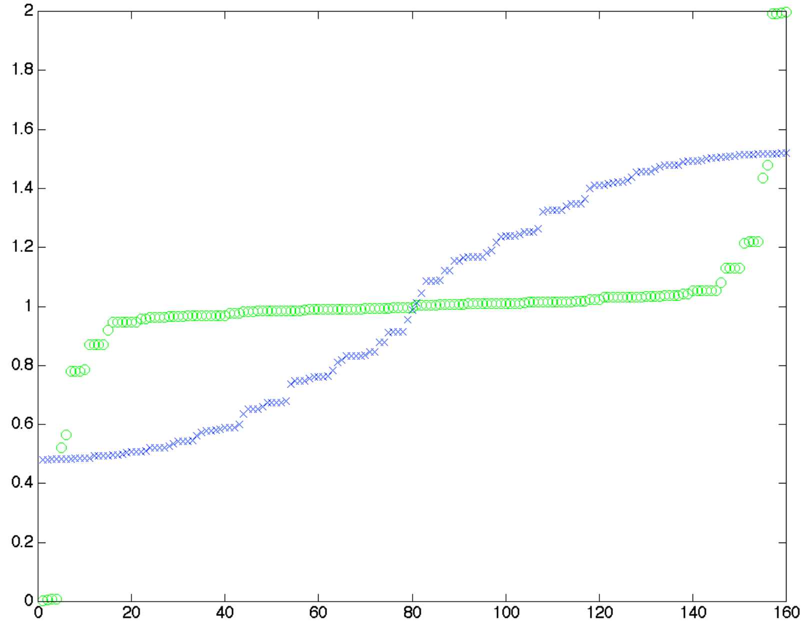


Eigenvalues of the interface problem (Genseberger, Sleijpen, van der Vorst, 2002)

The few very low eigenvalues are responsible for the plateau in the convergence of the Krylov method.
Two possible solutions: deflation techniques or more efficient ICs.
Deflation or Coarse Grid $\Rightarrow$ All the corresponding eigenvectors are needed (Morgan, Segal, Meijerink, Frank, Vuik, Saad, Erhel, Nabben, van der Vorst, Graham and Hagger. ...)

“Frozen” coefficients ICs



Eigenvalues of the interface problem (Genseberger, Sleijpen, van der Vorst, 2002)

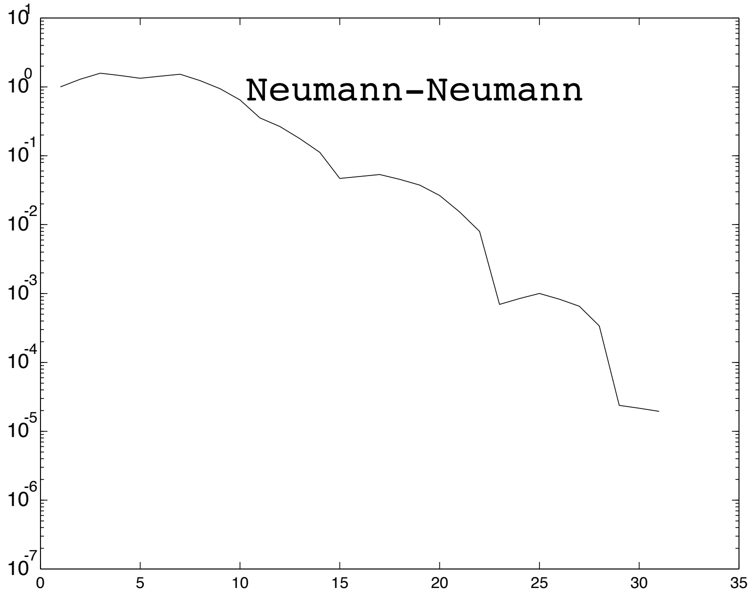
The few very low eigenvalues are responsible for the plateau in the convergence of the Krylov method.

Two possible solutions: deflation techniques or more efficient ICs.

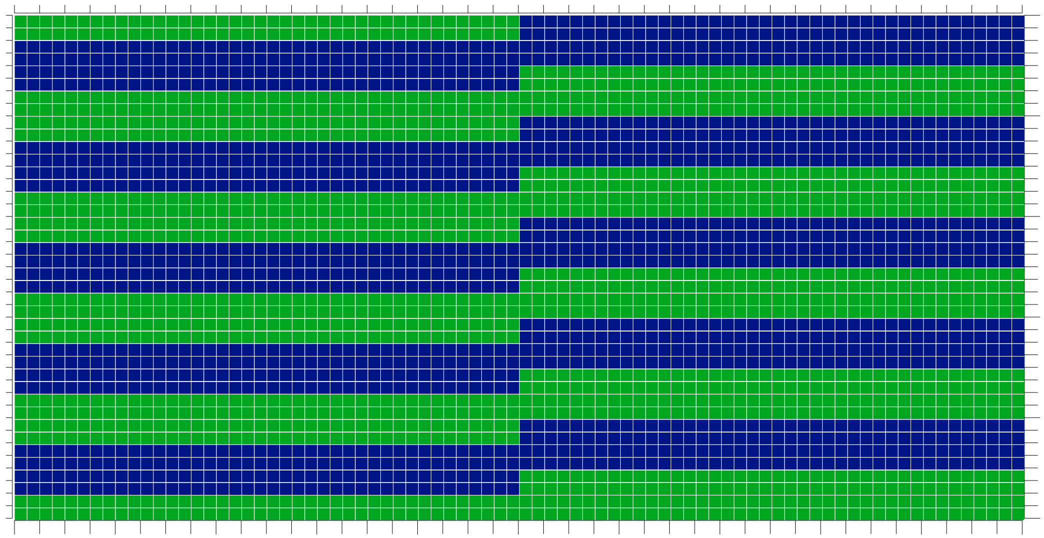
Deflation ⇒ All the corresponding eigenvectors are needed (Morgan, Segal, Meijerink, Frank, Vuik, Saad, Erhel, Nabben, van der Vorst, Graham and Hagger. ...)

Opt2 ICs ⇒ Only the two extremal eigenvalues of some auxiliary matrix are needed.

Generalized Neumann-Neumann/ FETI



residual vs. subdomain solve counts



Lythology

- Klawonn, Widlund, Dryja, Dual-Primal FETI methods for three-dimensional elliptic problems with heterogeneous coefficients (2002)
- Le Tallec. M. Vidrascu, Generalized Neumann-Neumann preconditioners for iterative substructuring (1996)
- Cowsar, J. Mandel, M.F. Wheeler, Balancing domain decomposition for mixed Finite elements (1995)

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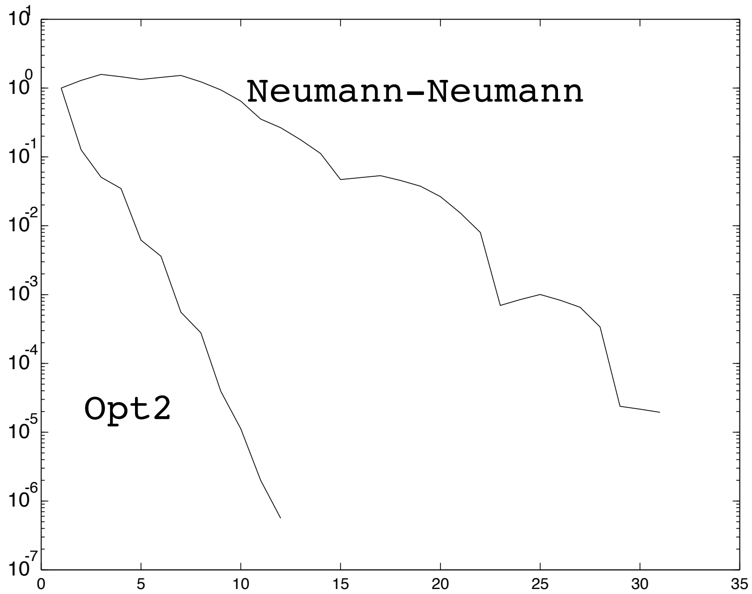

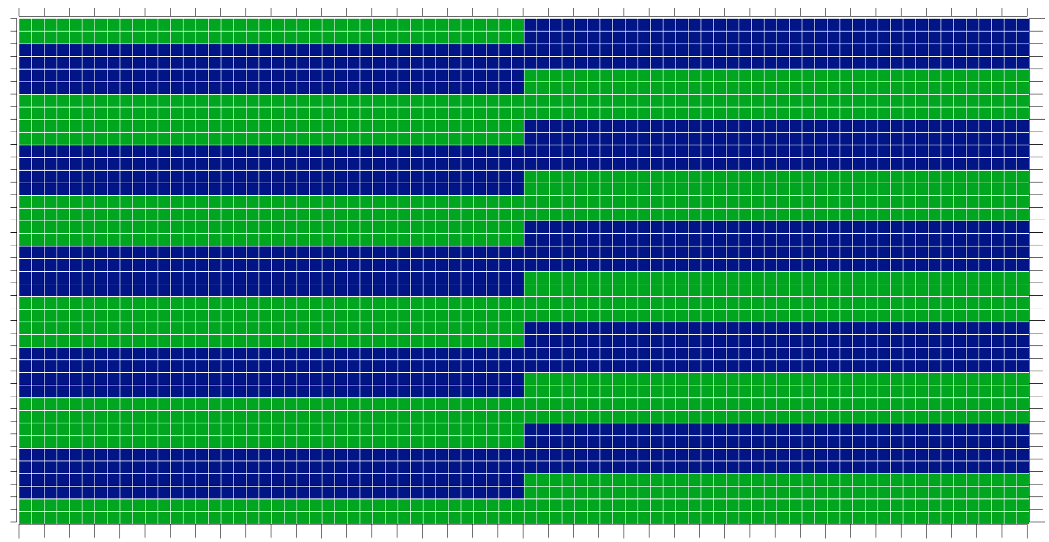

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   The equation

   "Frozen" coefficients interface conditions

   Deflation or coarse grid

   Generalized Neumann-Neumann/ FETI algorithm

   Optimized of Order 2 Interface Conditions

      Generalities on Optimized ICs

      Optimized ICs of order 0

      Optimized ICs of order 2
5. Conclusion

ICs for discontinuous coefficients at the continuous level

$$\mathcal{L}_i(P) = \eta_i P - \frac{\partial}{\partial x}(C_i \frac{\partial}{\partial x} P) - \frac{\partial}{\partial y}(\kappa_{i,y} \frac{\partial}{\partial y} P), \qquad i = 1, 2$$

with η_i, C_i and $\kappa_{i,y}$ independant of the x variable.
Let $\Omega_1 = \mathbb{R}^- \times]0, L[$, $\Omega_2 = \mathbb{R}^+ \times]0, L[$ and $\Gamma = \{0\} \times]0, L[$. Consider a Schwarz type method:

$$\begin{array}{ll}
\mathcal{L}_1(P_1^{n+1}) = f \quad \text{in } \Omega_1 & \mathcal{L}_2(P_2^{n+1}) = f \quad \text{in } \Omega_2 \\
P_1^{n+1} = 0 \text{ on } \partial\Omega_1 \backslash \Gamma & P_2^{n+1} = 0 \text{ on } \partial\Omega_2 \backslash \Gamma \\
(C_1 \partial_x + \Lambda_{ap,2})(P_1^{n+1}) & (C_2 \partial_x + \Lambda_{ap,1})(P_2^{n+1}) \\
= (-C_2 \partial_x + \Lambda_{ap,2})(P_2^n) \quad \text{on } \Gamma & = (-C_1 \partial_x + \Lambda_{ap,1})(P_1^n) \quad \text{on } \Gamma
\end{array}$$

We seek a partial differential approximation to the DtN_i map:

$$\Lambda_{ap,i} \simeq C_i^{1/2} \sqrt{C_i^{-1/2}(\eta_i - \frac{\partial}{\partial y}(\kappa_{i,y} \frac{\partial}{\partial y}))C_i^{-1/2}} C_i^{1/2} \; (:= \Lambda_i)$$

Optimized ICs for discontinuous coefficients at the semi discrete level

After a discretization in y, the operator $\mathcal{L}_i$ reads

$$\mathcal{L}_{i,h} := -\frac{\partial}{\partial x} C_i \frac{\partial}{\partial x} + B_i$$

where B_i is the finite volume discretization of

$$\eta_i(y) - \frac{\partial}{\partial y} \kappa_{i,y}(y) \frac{\partial}{\partial y}$$

These matrices are symmetric positive and definite.
The discrete Steklov-Poincaré (Dirichlet to Neumann) operator Λ_i, of subdomain Ω_i is

$$\Lambda_i = C_i^{1/2} (C_i^{-1/2} B_i C_i^{-1/2})^{1/2} C_i^{1/2}$$

We seek sparse approximations to Λ_i: $\Lambda_{ap,i}$.
An approximation by a diagonal matrix corresponds to a condition of order 0. An approximation by a tridiagonal matrix corresponds to a condition of order 2.

Optimized ICs

Both subdomains have the same lythology.

Lemma 2 (Gensberger & al.) *We assume that Λ_{ap} is a SPD matrix.*
Let $\rho_{Sc}(\Lambda_{ap})$ be the convergence rate of the Schwarz algorithm.
We have $\rho_{Sc}(\Lambda_{ap}) = \|(\Lambda - \Lambda_{ap})(\Lambda + \Lambda_{ap})^{-1}\|_2 < 1$
Moreover, the matrix $Sub(\Lambda_{ap})$ of the interface problem has real eigenvalues in $(0, 2)$ symmetric w.r.t one and

$$\kappa_{eff}(Sub(\Lambda_{ap})) = \frac{1 + \rho_{Sc}(\Lambda_{ap})}{1 - \rho_{Sc}(\Lambda_{ap})}$$

where $\kappa_{eff}(A) = \lambda_M(A)/\lambda_m(A)$.

Optimizing the interface condition for the solving of the interface problem by a Jacobi method (Schwarz method) or by a GMRES method are equivalent.

Optimized ICs for discontinuous coefficients

Lemma 3 *Let* Λ_{ap} *be a SPD matrix and* λ_M *(resp.* λ_m*) be the largest (resp. smallest) eigenvalue of* $\Lambda_{ap}^{-1/2}\Lambda\Lambda_{ap}^{-1/2}$.
Then,

$$\min_{\beta \in \mathbb{R}^+} \rho_{Sc}(\beta\Lambda_{ap}) = \rho_{Sc}(\beta_{opt}\Lambda_{ap})) = \frac{\sqrt{\frac{\lambda_M}{\lambda_m}} - 1}{\sqrt{\frac{\lambda_M}{\lambda_m}} + 1}$$

where

$$\beta_{opt} = \sqrt{\lambda_M \lambda_m}$$

We have to find sparse matrices D that minimize the condition number of $D^{-1/2}\Lambda D^{-1/2}$ and then take $\Lambda_{ap} = \beta_{opt} D$.

Optimized zeroth order ICs for discontinuous coefficients

In order to use the previous Lemma, we have to find the best diagonal matrix D that minimizes the condition number of $D^{-1/2}\Lambda D^{-1/2}$.

Theorem 1 (van der Sluis, 1969) *If F is SPD matrix, then*

$$\begin{aligned} \min_{D\in\mathcal{D}} \kappa(D^{-1/2}FD^{-1/2}) \leq \quad & \kappa(diag(F)^{-1/2}F diag(F)^{-1/2}) \\ & \leq m\,.\,\min_{D\in\mathcal{D}} \kappa(D^{-1/2}FD^{-1/2}) \end{aligned}$$

where $\mathcal{D} = \{positive\ definite\ diagonal\ matrices\}$ and m is the maximum number of nonzeros in any row of F.

We are led to take

$$D := diag(\Lambda) = diag(C^{1/2}\,(C^{-1/2}BC^{-1/2})^{1/2}\,C^{1/2}),$$

$$\beta_{opt} = \sqrt{\lambda_M(D^{-1/2}\Lambda D^{-1/2})\lambda_m(D^{-1/2}\Lambda D^{-1/2})}$$

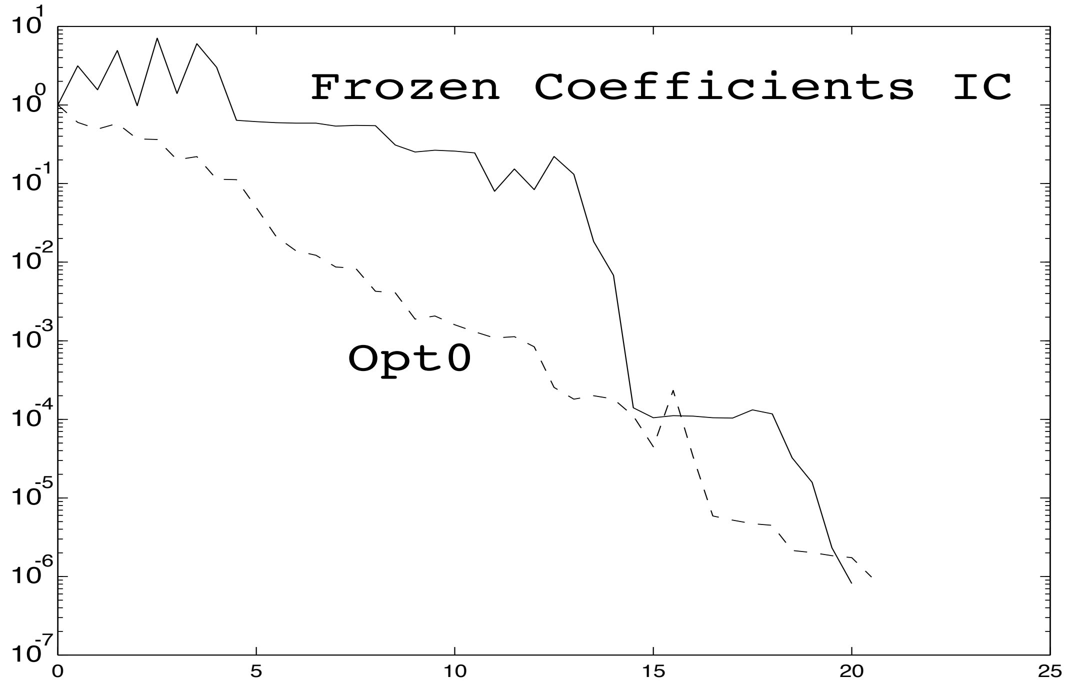
and $$\Lambda_{ap} = \beta_{opt} D$$

Practical Optimized zeroth order ICs

Let $A := C^{-1/2}BC^{-1/2}$ then, $D := C^{1/2}diag(A^{1/2})C^{1/2}$ is approximated by $\tilde{D} := C^{1/2}diag(A)^{1/2}C^{1/2}$

$$\tilde{\beta}_{opt} = (\lambda_M(diag(A)^{-1/2}A\,diag(A)^{-1/2})^{1/2}\lambda_m(diag(A)^{-1/2}A\,diag(A)^{-1/2})^{1/2})^{1/2}$$

and $$\Lambda_{ap} = \tilde{\beta}_{opt}\tilde{D}$$



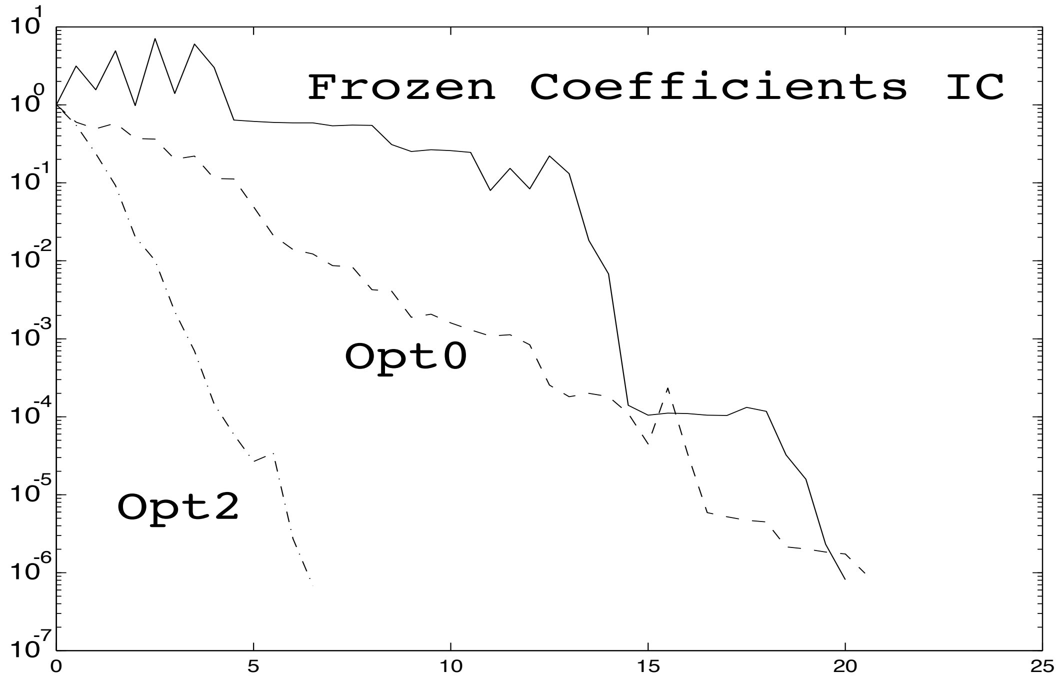
residual vs. iteration count

Practical Optimized zeroth order ICs

Let $A := C^{-1/2} B C^{-1/2}$ and $\tilde{D} := C^{1/2} diag(A)^{1/2} C^{1/2}$,

$$\tilde{\beta}_{opt} = (\lambda_M (diag(A)^{-1/2} A\, diag(A)^{-1/2})^{1/2} \lambda_m (diag(A)^{-1/2} A\, diag(A)^{-1/2})^{1/2})^{1/2}$$

and
$$\Lambda_{ap} = \tilde{\beta}_{opt} \tilde{D}$$



residual vs. iteration count

Optimized ICs of order two

Combination of two zeroth order ICs thanks to Higdon's trick (Math. of Comp., 1986)

$$\mathcal{Q} := (C\frac{\partial}{\partial x} + \beta_1 \tilde{D})(C\frac{\partial}{\partial x} + \beta_2 \tilde{D})$$

for some positive parameters β_1, β_2.
This product yields a second order derivative w.r.t x the normal direction to the interface:

$$\mathcal{Q} := C\frac{\partial}{\partial x}(C\frac{\partial}{\partial x}) + (\beta_1 + \beta_2)\tilde{D}C\frac{\partial}{\partial x} + \beta_1\beta_2\tilde{D}^2$$

By using the operator $\mathcal{L}_h = -\partial_x C \partial_x + B$ this second order derivative can be replaced by

$$CB$$

so that condition $\mathcal{Q}$ is equivalent to

$$\mathcal{Q} := CB + (\beta_1 + \beta_2)\tilde{D}C\frac{\partial}{\partial x} + \beta_1\beta_2\tilde{D}^2.$$

Optimized second order ICs

We still have to write this condition in the form

$$C\frac{\partial}{\partial x} + \Lambda_{ap,2}$$

for some operator $\Lambda_{ap,2}$. Since interface conditions are equivalent up to the left composition with any invertible operator acting along the interface, we obtain an equivalent condition $\mathcal{R}$ by left multiplying $\mathcal{Q}$ by the inverse of $(\beta_1 + \beta_2)\tilde{D}$:

$$\mathcal{R} := C\frac{\partial}{\partial x} + C^{1/2}\frac{\tilde{D}^{-1}A + \beta_1\beta_2\tilde{D}}{\beta_1 + \beta_2}C^{1/2} \qquad (1)$$

In other words, we choose to approximate Λ by

$$\Lambda_{ap,\beta_1,\beta_2} := C^{1/2}\frac{\tilde{D}^{-1}A + \beta_1\beta_2\tilde{D}}{\beta_1 + \beta_2}C^{1/2} \qquad (2)$$

with $\beta_1, \beta_2 > 0$.

Optimized second order ICs: best β_1, β_2

Based on Wachspress optimization of ADI methods, we have

Theorem 2 *Suppose matrices $\tilde{D}$ and $A^{1/2}$ commute. Let $\lambda_m := \lambda_{min}(\tilde{D}^{-1}A^{1/2})$ and $\lambda_M := \lambda_{max}(\tilde{D}^{-1}A^{1/2})$. The choice*

$$\beta_{1,opt}\beta_{2,opt} = \lambda_m \lambda_M \tag{3}$$

$$\beta_{1,opt} + \beta_{2,opt} = \left(\min_{\lambda \in Sp(\tilde{D}^{-1}A^{1/2})} (\lambda + \frac{\lambda_m \lambda_M}{\lambda}) (\lambda_m + \lambda_M) \right)^{1/2} \tag{4}$$

is optimal in the sense that:

$$\min_{\beta_1 \in \mathbb{R}^+, \beta_2 \in \mathbb{R}^+} \kappa(Sub(\Lambda_{ap,\beta_1,\beta_2})) = \kappa(Sub(\Lambda_{ap,\beta_{1,opt},\beta_{2,opt}}))$$

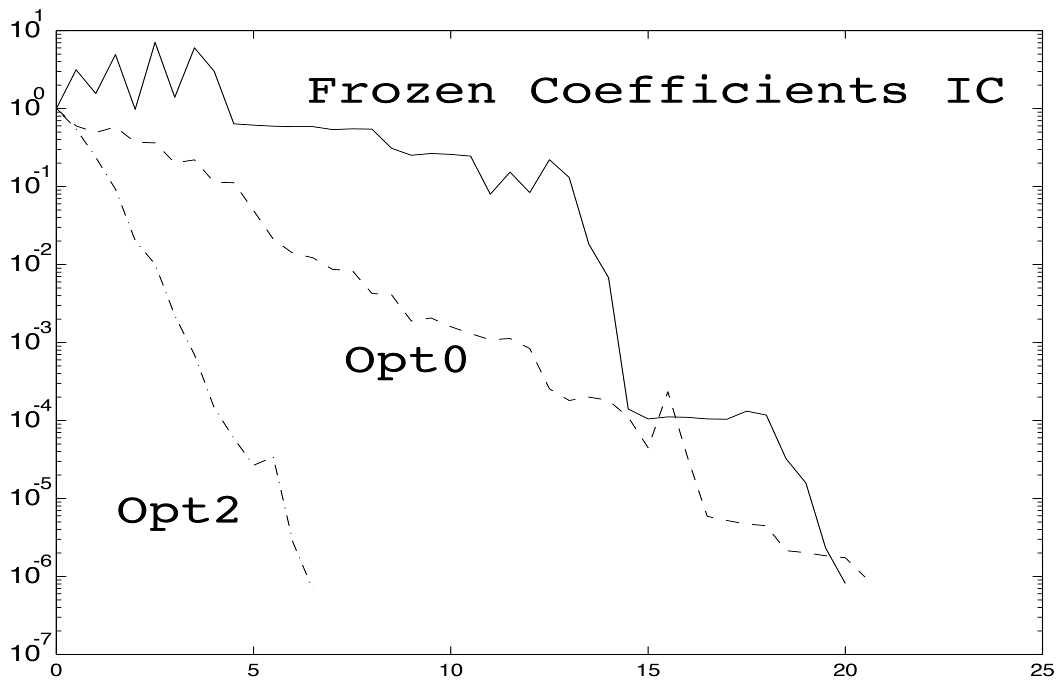
We have a bound on the condition number

$$\kappa(Sub(\Lambda_{ap,\beta_{1,opt},\beta_{2,opt}})) \leq \frac{1}{\sqrt{2}} \left(\sqrt{\frac{\lambda_M}{\lambda_m}} + \sqrt{\frac{\lambda_m}{\lambda_M}} \right)^{1/2}$$

In practice, $\lambda_{m,M} \simeq \lambda_{m,M}(D^{-2}A)^{1/2}$.

Numerical results – Different Subdomains

This approach has been generalized to the "fully" discrete case (no more ∂_x).
Optimization parameters obtained for two identical subdomains are used when the subdomains are different.



residual vs. iteration count

Numerical results

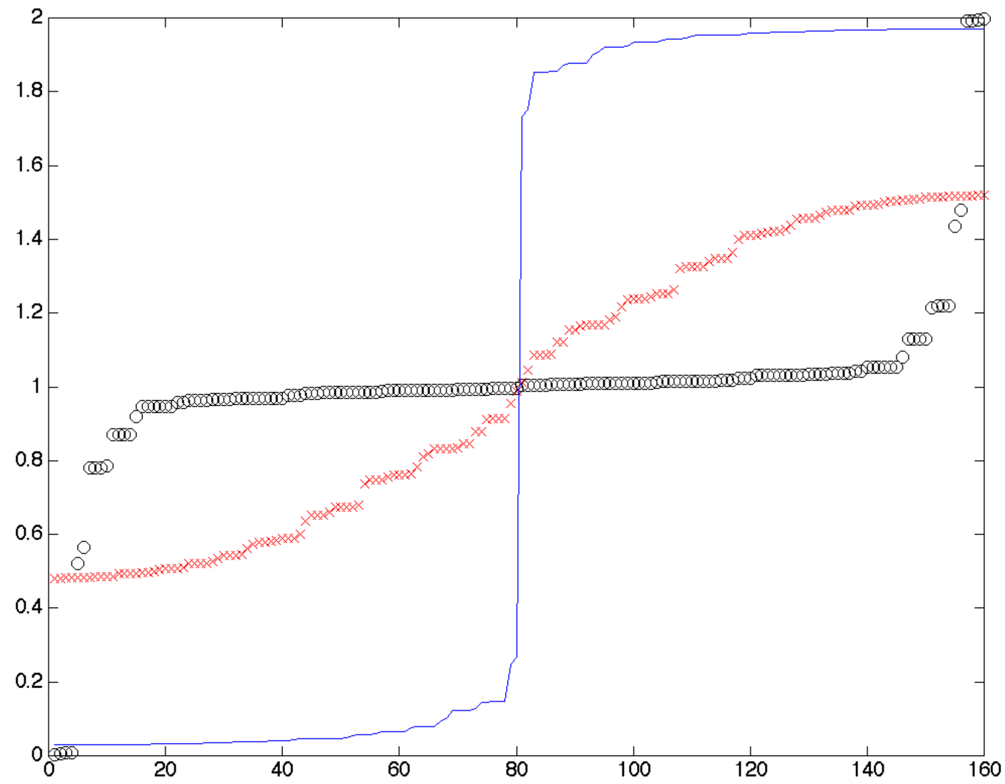


Figure 1: Eigenvalues of the interface problem: circle: "Frozen" coefficients ICs, solid line: opt0, cross: opt2

Numerical results – Different Subdomains

Table 1: Results for highly heterogeneous and anistropic problems

ny		40	80	160	320
(opt0)	#sub solv	32	40	48	56
	$\|\lambda\|_M / real(\lambda)_m$	43.5	65.1	94.1	1.3e+2
(opt2)	#sub solv	13	15	15	16
	$\|\lambda\|_M / real(\lambda)_m$	4.6	5.7	6.8	8.2
(Neumann-	#sub solv	24	24	24	28
Neumann)	λ_M / λ_m	3.0e+2	4.2e+2	6.2e+2	9.6e+2

Heterogeneous media

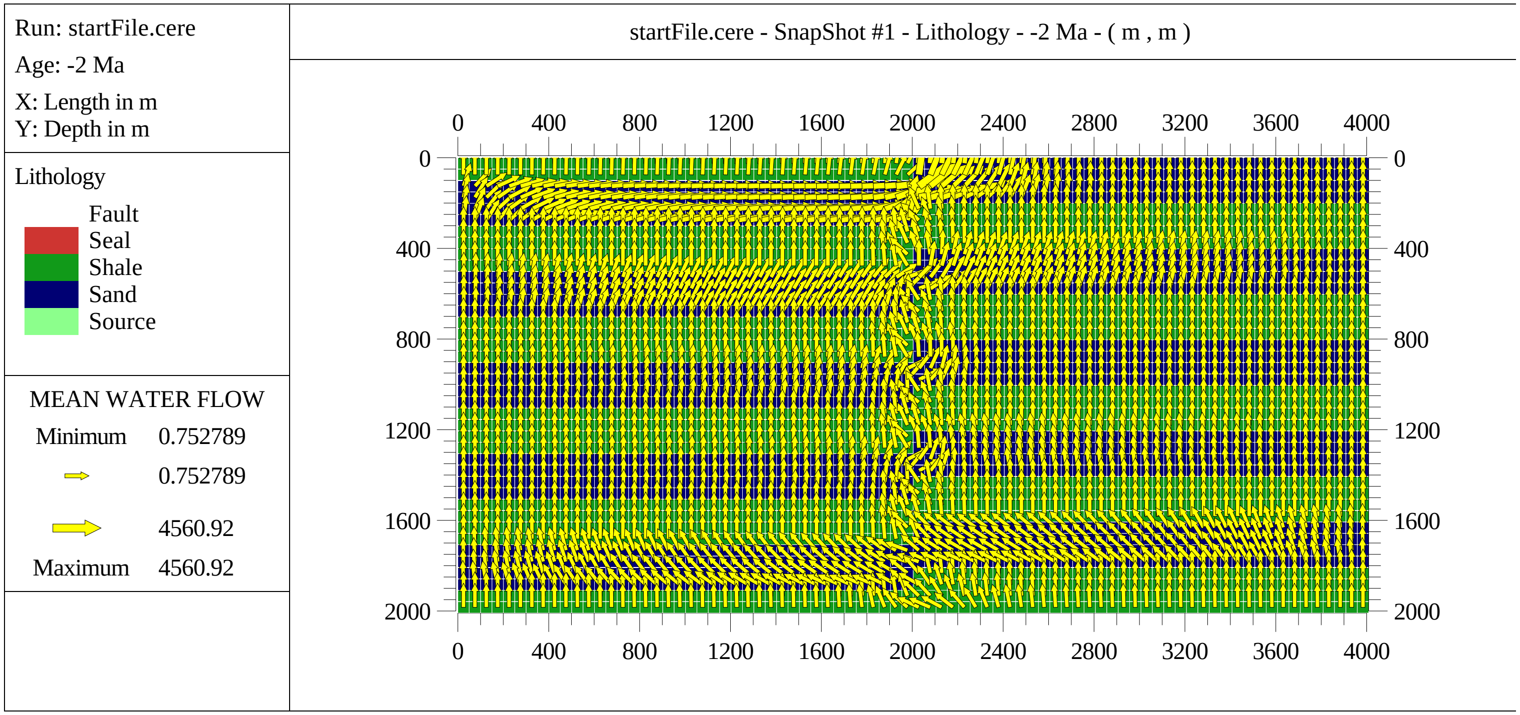


Figure 2: Lithology and fluxes for a bounded domain simulation These results have been obtainded with a prototype code developped at IFP

Iteration counts: 11 iterations for the opt2 interface conditions and 21 for Robin interface conditions.

Conclusion

For problems with highly anisotropic and discontinuous coefficients, plateaus in the convergence of Krylov methods exist even when using “good” preconditioners. A classical remedy is to use deflated Krylov methods or fine enough coarse grids.
We have developped a new algebraic approach in the DDM framework that solves the problem.

- Compared to deflation, only two extreme eigenvalues have to be computed.
- Can be used on non matching grids with FE or FV methods.
- Unsymmetric problems: joint work with L. Gerardo-Giorda

Prospects

- Extension to multiple intersections: cross points,... (in progress with F. Nier and C. Chniti)
- Extension to systems of PDE’s (cf. V. Dolean on compressible Euler equations)

Thanks !