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NASH EQUILIBRIUM CONDITIONS — EXTENSIONS OF SOME CLASSICAL THEOREMS

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Consider a non-cooperative game $\Gamma = \langle N, \{X_p\}_{p\in N}, \{f_p(\mathbf{x})\}_{p\in N}\rangle$, where
$N = \{1, 2, ..., n\}$ is a set of players,
$X_p = \{x^p \in R^{n_p} : g_i^p(x^p) \leq 0, i = \overline{1, m_p}, h_i^p(x^p) = 0, i = \overline{1, l_p}, x \in M_p\}$ is a set of strategies of player $p \in N$, $l_p, m_p, n_p < +\infty$, $p \in N$,
$g_i^p(x^p), h_i^p(x^p)$ are functions defined on M_p, $p \in N$,
$f_p(\mathbf{x})$ is a player's $p \in N$ payoff function defined on $X = \underset{p\in N}{\times} X_p$.

Suppose that the players minimize the cost of their payoff functions.

Definition [1]. *The outcome $\hat{x} \in X$ of the game is a Nash equilibrium (NE) if $f_p(x^p, \hat{x}^{-p}) \geq f_p(\hat{x}^p, \hat{x}^{-p}), \forall x^p \in X_p$, $\forall p \in N$.* It is well known that the convex continuous compact games have NE [1].

Let $L_p(x, u^p, v^p) = f_p(x) + \sum_{i=1}^{m_p} u_i^p g_i^p(x^p) + \sum_{i=1}^{l_p} v_i^p h_i^p(x^p)$ be the Lagrange function for the player $p \in N$, where $u^p = (u_1^p, u_2^p, ..., u_{m_p}^p)$, $v^p = (v_1^p, v_2^p, ..., v_{m_p}^p)$ are the Lagrange multipliers. Let $L = (L_1, ..., L_n)$ be the Lagrange vector-function.

Definition. $(\hat{x}, \hat{u}, \hat{v}) \in X \times R_{\geq}^{m_1} \times R^{l_1} \times ... \times R_{\geq}^{m_n} \times R^{l_n}$ *is a saddle point for L if* $L_p(\hat{\mathbf{x}}, u^p, v^p) \leq L_p(\hat{x}, \hat{u}^p, \hat{v}^p) \leq L_p(x^p, \hat{x}^{-p}, \hat{u}^p, \hat{v}^p)$ $\forall$ $(x^p, u^p, v^p) \in M_p \times R_{\geq}^{m_p} \times R^{l_p}$, $p \in N$.

Theorem 1. *If $(\hat{x}, \hat{u}, \hat{v})$ is a saddle point for L, then $\hat{x}$ is a Nash equilibrium for Γ.*

Consider Γ with $X_p = \{x^p \in R^{n_p} : g_i^p(x^p) \leq 0, i = \overline{1, m_p}\}$ where $g_i^p(x^p), i = \overline{1, m_p}$ are convex on R^{n_p}. Let, also, $f_p(x^p, x^{-p}), p = \overline{1, n}$ be convex and continuous on X_p for every fixed $x^{-p} \in X_{-p}$. $L_p(x, u^p) = f_p(x) + \sum_{i=1}^{m_p} u_i^p g_i^p(x^p)$.

Theorem 2. *Let every X_p be compact and satisfy the Slater regularity condition. The outcome $\hat{x} \in X$ is a NE for Γ if and only if there exist $\hat{u}^p \geq 0, p = \overline{1, n}$ so that*

$(\hat{x}, \hat{u})$ *is a saddle point for* L.

Theorem 3. *Suppose that* $f_p(x), g_i^p(x^p), i = \overline{1, m_p},\ p = \overline{1, n}$ *are differentiable on* $\hat{x}$*, every* X_p *is compact and satisfies the Slater regularity condition. The outcome* $\hat{x} \in X$ *is a NE for* Γ *if and only if there exist* $\hat{u}^p \geq 0, p = \overline{1, n}$ *so that*

$$\frac{\partial L_p(\hat{x}, \hat{u}^1, ..., \hat{u}^p)}{\partial x_j^p} = 0, j = \overline{1, n_p}, \qquad \hat{u}_i^p g_i^p(\hat{x}^p) = 0, i = \overline{1, m_p}, \quad p = \overline{1, n}.$$

Theorems 1–3 are extensions on normal form strategic games of the well known theorems for matrix games. Theorem 3 is a Kuhn-Tucker type theorem for normal form strategic games. Other theorems of this kind are formulated and proved.

Games with vector payoffs are examined. The set of all Pareto-Nash equilibria may be defined as *intersection of the graphs of efficient response mappings*, analogically with definition of the Nash equilibria set for strategic form games [2,3]. Pareto-Nash equilibrium conditions are formulated and proved.

Bibliographie

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