Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 184 (2007) 1015–1031
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

Unconditional stability of parallel difference schemes with second order accuracy for parabolic equation ☆

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Abstract

In this paper we investigate the parallel difference schemes of parabolic equation, in particular, two kinds of difference schemes with intrinsic parallelism are constructed. Firstly we combine the values of previous two time levels at the interface points to get the (Dirichlet) boundary condition for the sub-domain problems. Then the values in the sub-domains are calculated by fully implicit scheme. And then finally the values at the interface points are computed by fully implicit scheme. The unconditional stability of these schemes is proved, and the convergence rate of second order is also obtained. Numerical results are presented to examine the accuracy, stability and parallelism of the parallel schemes.

Keywords: Parabolic equation; Parallel difference; Unconditional stability; Convergence; Second-order accuracy

1. Introduction

There is rich literature on the parallel difference schemes for the parabolic equation (see [1–10]). Explicit schemes are often naturally parallel and also easy to implement, but they usually require small time steps because of stability constraints. Implicit schemes are necessary for finding steady state solution or computing slowly unsteady problems where one needs to march with large time steps. However, implicit schemes are not inherently parallel.

The alternating schemes in [1–5] use the explicit scheme and implicit scheme alternately in the time and space direction, which can implement the parallel computation and are unconditionally stable, i.e., for *any* positive constant C, when $\lambda \leqslant C$, the scheme is stable. For the heat equation $u_t = u_{xx}$ the classic explicit scheme is not unconditional stable since C cannot be taken larger than $\frac{1}{2}$, however the classic implicit scheme and some alternating schemes are unconditional stable. Note that these alternating schemes can not be implemented directly by making use of the original sequential codes.

☆ The project is supported by the Special Funds for Major State Basic Research Projects 2005CB321703, the National Nature Science Foundation of China (No. 10476002, 60533020).
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doi:10.1016/j.amc.2006.07.003

Domain decomposition is a powerful tool for devising parallel PDE methods. Much of the work on domain decomposition has been directed at elliptic equation (see [11]). In this paper, we present a finite difference method which utilizes domain decomposition to allow us to divide the work of solving the heat equation. The method differs from the methods mentioned in [11] that it is noniterative. It can be used to divide the global problem into smaller sub-domain problems, which can be solved in parallel. The major difficulties with such procedures involve defining values on the sub-domain boundaries and piecing the solutions together into a reasonable approximation to the true solution. Once the interface values are available, the global problem is fully decoupled and thus be computed in parallel. A domain decomposition scheme was proposed in [6], where instead of using the same spacing h as for the interior points where the implicit scheme is applied, a larger spacing H_D is used at each interface points where the explicit scheme is applied. Due to stability and accuracy requirement, the method does not lead to satisfactory parallel efficiency, although the method can be implemented with little extra effort using the original sequential codes. there also have some other schemes with domain decomposition in [7,8]. These schemes are generally of second order global accuracy in space, i.e., $O(\Delta t + h^2)$, but all of them are conditionally stable.

The unconditional stable scheme is desirable in solving some application problems. Some unconditionally stable schemes were proposed in [9], which firstly take the values of previous time step as the boundary condition, then solve the sub-domain problems by implicit scheme in parallel, and then finally update the interface values by implicit scheme. These schemes can be easily implemented using the original sequential codes, but they are only of one order global accuracy in space. In order to improve the global accuracy, the parallel iterative difference schemes based on interface correction were proposed in [10]. But it can not assure second order global accuracy in theory.

Up until now the available parallel difference schemes at least have one of the following defects: (1) they are conditionally stable (e.g. [6–8]), i.e., there exists a positive constant $C < +\infty$ such that the schemes are unstable when $\lambda > C$; (2) the accuracy of these parallel schemes is lower than fully implicit scheme (e.g. [1–5,9]); (3) the design of algorithm is complex (e.g. [1–5,10]), and therefore the scheme can not be implemented with little extra effort with the original sequential codes.

In this paper, we propose two kinds of parallel difference scheme with intrinsic parallelism. The resulting schemes are of second order global accuracy in space and unconditionally stable. The principle of the method lies mainly in the following steps. First we get the inner boundary condition by combining the values of previous two time levels at the interface points, then compute the values in the sub-domain by fully implicit scheme, and then finally update the interface values by fully implicit scheme. When the interface values are obtained by the linear combination, the global problem is fully decoupled and can thus be computed in parallel. No iterations between sub-domains are necessary. Hence, our method should be nearly optimal. The calculations of updating the interface values are explicit, since the values adjacent to the interface points have been obtained. The time needed to update the interface values is very small relative to the time needed to solve sub-domain problems. There only need to pass information between neighboring sub-domains and no global communication is necessary. Thus, the parallel algorithm is fully scalable. Furthermore, the method is easy to incorporate into existing implicit codes.

The rest of this paper is organized as follows. In the next section, we propose two kinds of parallel difference scheme for a one-space-dimensional problem. In Section 3, we prove the stable and convergence result for these schemes. The resulting schemes are unconditionally stable and convergence in the sense of discrete H^1 norm, and of second order accuracy in space, i.e., $O(\Delta t + h^2)$. In Sections 4 and 5, we extend the one-dimensional results to two space dimensions, and prove the stability and convergence results. In Section 6, we examine numerically the accuracy, stability and parallelism of the scheme on certain test problems. The numerical results verify the theoretical results. Moreover it is shown that the super-linear speedup is achieved. In Section 7, we give some generalizations and conclusions.

2. Construction of one-dimensional schemes

2.1. Problem and notation

Let $U(x,t)$ be the solution of the following model problem with the initial and boundary conditions.

$$U_t = U_{xx}, \quad (x,t) \in (0,1) \times (0,T], \tag{2.1}$$
$$U(0,t) = U(1,t) = 0, \quad t \in (0,T], \tag{2.2}$$
$$U(x,0) = U_0(x), \quad x \in (0,1). \tag{2.3}$$

The initial and boundary data satisfy the compatibility conditions $U_0(0) = U_0(1) = 0$.

Divide the domain $(0,1)\times(0,T]$ by $x_i = ih$, $t^n = n\tau$. Let $\tau = T/N$, $h = 1/J$, $\lambda = \tau/h^2$. For a function $\phi(x,t)$ defined at the mesh points (x_i,t^n), let $\phi_i^n = \phi(x_i,t^n)$. Define the difference operators

$$\delta u_j^n = \frac{1}{h}(u_{j+1}^n - u_j^n), \quad \delta^2 u_j^n = \frac{1}{h}(\delta u_j^n - \delta u_{j-1}^n), \quad \Delta_\tau u_j^{n+1} = \frac{1}{\tau}(u_j^{n+1} - u_j^n).$$

The discrete norms are defined as follows:

$$\|u_h^n\|_2^2 = \sum_{j=0}^{J} |u_j^n|^2 h, \quad \|u_h^n\|_\infty^2 = \max_{0\leqslant j\leqslant J} |u_j^n|^2.$$

We will construct following two kinds of difference scheme with intrinsic parallelism.

2.2. Parallel scheme 1 (S1)

Let k be a positive integer such that $2 < k < J-2$, assume u_j^{n-1} and u_j^n are given. At first compute the values of $\{u_j^{n+1} | j \neq k\}$ by the following equations:

$$\Delta_\tau u_j^{n+1} = \delta^2 u_j^{n+1}, \quad j = 1,2,\ldots,k-2,k+2,\ldots,J-1, \tag{2.4}$$
$$\Delta_\tau u_{k-1}^{n+1} = \frac{1}{h^2}((2u_k^n - u_k^{n-1}) - u_{k-1}^{n+1} - (u_{k-1}^{n+1} - u_{k-2}^{n+1})), \tag{2.5}$$
$$\Delta_\tau u_{k+1}^{n+1} = \frac{1}{h^2}(u_{k+2}^{n+1} - u_{k+1}^{n+1} - (u_{k+1}^{n+1} - (2u_k^n - u_k^{n-1}))). \tag{2.6}$$

Then compute the value of u_k^{n+1} by implicit scheme

$$\Delta_\tau u_k^{n+1} = \delta^2 u_k^{n+1}. \tag{2.7}$$

In Eqs. (2.4)–(2.7), (2.4) and (2.7) are fully implicit scheme, (2.5) and (2.6) are obtained by replacing u_k^{n+1} with $2u_k^n - u_k^{n-1}$ in a stencil of implicit scheme at $j = k-1$ and $k+1$, respectively, therefore, the global problem is fully decoupled and the values of $\{u_j^{n+1} | j < k\}$ and $\{u_j^{n+1} | j > k\}$ can thus be computed in parallel. Then we can compute explicitly the values of u_k^{n+1} by implicit scheme (2.7). It is obvious that the truncation error of the scheme (2.4)–(2.7) is $O(\tau + h^2)$. Eqs. (2.5) and (2.6) need the values of previous two time levels at interface points, so we use implicit scheme at the first time level, i.e.

$$\Delta_\tau u_j^1 = \delta^2 u_j^1, \quad j = 1,2,\ldots,J-1. \tag{2.8}$$

Eq. (2.5) is equivalent to

$$\begin{aligned}\Delta_\tau u_{k-1}^{n+1} &= \frac{1}{h^2}(u_k^n - u_{k-1}^{n+1} - (u_{k-1}^{n+1} - u_{k-2}^{n+1})) + \lambda\Delta_\tau u_k^n \\ &= \frac{1}{h^2}(u_k^n - u_{k-1}^{n+1} - (u_{k-1}^{n+1} - u_{k-2}^{n+1})) + \lambda\delta^2 u_k^n.\end{aligned}$$

Obviously it only need the values of previous time level.

Eqs. (2.4)–(2.7) are equivalent to the following equations:

$$\Delta_\tau u_j^{n+1} = \delta^2 u_j^{n+1}, \quad j = 1,2,\ldots,k-2,k,k+2,\ldots,J-1, \tag{2.9}$$
$$\Delta_\tau u_{k-1}^{n+1} = \delta^2 u_{k-1}^{n+1} - \lambda\Delta_\tau u_k^{n+1} + \lambda\Delta_\tau u_k^n, \tag{2.10}$$
$$\Delta_\tau u_{k+1}^{n+1} = \delta^2 u_{k+1}^{n+1} - \lambda\Delta_\tau u_k^{n+1} + \lambda\Delta_\tau u_k^n. \tag{2.11}$$

2.3. Parallel scheme 2 (S2)

Let k be a positive integer such that $2 < k < J-2$, and assume $\{u_j^{n-1}\}$ and $\{u_j^n\}$ are given. At first compute the values of $\{u_j^{n+1} | j \neq k\}$ by the following equations:

$$\Delta_\tau u_j^{n+1} = \delta^2 u_j^{n+1}, \quad j = 1,2,\ldots,k-1,k+1,\ldots,J-1, \tag{2.12}$$

$$\Delta_\tau \bar{u}_k^{n+1} = \frac{1}{h^2}((2u_{k+1}^n - u_{k+1}^{n-1}) - \bar{u}_k^{n+1} - (\bar{u}_k^{n+1} - u_{k-1}^{n+1})), \tag{2.13}$$

$$\Delta_\tau \tilde{u}_k^{n+1} = \frac{1}{h^2}(u_{k+1}^{n+1} - \tilde{u}_k^{n+1} - (\tilde{u}_k^{n+1} - (2u_{k-1}^n - u_{k-1}^{n-1}))), \tag{2.14}$$

where

$$\delta^2 u_{k-1}^{n+1} = \frac{1}{h^2}(\bar{u}_k^{n+1} - u_{k-1}^{n+1} - (u_{k-1}^{n+1} - u_{k-2}^{n+1})),$$

$$\delta^2 u_{k+1}^{n+1} = \frac{1}{h^2}(u_{k+2}^{n+1} - u_{k+1}^{n+1} - (u_{k+1}^{n+1} - \bar{u}_k^{n+1})).$$

There are different values at the mesh point $x = x_k$ in different subdomains. In order to make the value unique, some average strategies must be taken. For simplicity, we take the arithmetic average of the two values as the value at the mesh point $x = x_k$, i.e.

$$u_k^{n+1} = \frac{1}{2}(\bar{u}_k^{n+1} + \tilde{u}_k^{n+1}). \tag{2.15}$$

In the scheme (2.12)–(2.15), (2.12) is fully implicit, while (2.13) and (2.14) are obtained by replacing u_{k+1}^{n+1} with $2u_{k+1}^n - u_{k+1}^{n-1}$, u_{k-1}^{n-1} with $2u_{k-1}^n - u_{k-1}^{n-1}$ in a fully implicit stencil at $j = k$, respectively. Therefore, the global problem is fully decoupled, and we can simultaneously compute the values of u_j^{n+1} in the subdomains $j \leqslant k$ and $j \geqslant k$ by Eqs. (2.12), (2.13) and (2.12), (2.14), respectively. And then compute the value at the mesh point $j = k$ by Eq. (2.15).

For $n > 0$, Eqs. (2.12)–(2.15) are equivalent to

$$\Delta_\tau u_j^{n+1} = \delta^2 u_j^{n+1}, \quad j = 1,2,\ldots,k-1,k+1,\ldots,J-1,$$
$$\Delta_\tau \bar{u}_k^{n+1} = \delta^2 \bar{u}_k^{n+1} - \lambda\Delta_\tau u_{k+1}^{n+1} + \lambda\Delta_\tau u_{k+1}^n,$$
$$\Delta_\tau \tilde{u}_k^{n+1} = \delta^2 \tilde{u}_k^{n+1} - \lambda\Delta_\tau u_{k-1}^{n+1} + \lambda\Delta_\tau u_{k-1}^n,$$
$$u_k^{n+1} = \frac{1}{2}(\bar{u}_k^{n+1} + \tilde{u}_k^{n+1}).$$

And it follows:

$$\Delta_\tau u_j^{n+1} = \delta^2 u_j^{n+1}, \quad j = 1,2,\ldots,k-1,k+1,\ldots,J-1, \tag{2.16}$$

$$\Delta_\tau u_k^{n+1} = \delta^2 u_k^{n+1} - \frac{1}{2}\lambda\Delta_\tau u_{k+1}^{n+1} + \frac{1}{2}\lambda\Delta_\tau u_{k+1}^n - \frac{1}{2}\lambda\Delta_\tau u_{k-1}^{n+1} + \frac{1}{2}\lambda\Delta_\tau u_{k-1}^n. \tag{2.17}$$

It is obvious that the truncation error of the scheme is $O(\tau + h^2)$. For $n = 0$, The implicit scheme (2.8) be used.

3. Unconditional stability and convergence of one-dimensional schemes

3.1. Theorem of stability and convergence

We have the following theorem of stability and convergence for the scheme constructed in previous section.

Theorem 3.1. *Parallel difference schemes* (S1) *and* (S2) *are unconditional stable in the sense of discrete H^1 norm, i.e., for any given $\lambda > 0$,*

$$\|\delta u_h^n\|_2^2 \leqslant C\|\delta u_h^0\|_2^2, \quad \forall n = 1,2,\ldots,N. \tag{3.1}$$

where C is a positive constant.

Let $e_j^n = U_j^n - u_j^n$. Then the approximate solution satisfies the following a priori error estimate:

Theorem 3.2. *Parallel schemes* (*S*1) *and* (*S*2) *are unconditional convergent*, *i.e.*, *for any given* $\lambda > 0$,

$$\|\delta e_h^n\|_2 \leqslant C(\tau + h^2), \quad \forall n = 1, 2, \ldots, N. \tag{3.2}$$

where C is a positive constant.

The proof of the Theorem relies on the following discrete Green formula.

Lemma 3.3. *Let* u_j *and* v_j *be the discrete function defined at* $\{x_j | j = 0, \ldots, J\}$, *then*

$$\sum_{j=0}^{J-1} u_j(v_{j+1} - v_j) = -\sum_{j=1}^{J-1}(u_j - u_{j-1})v_j - u_0v_0 + u_{J-1}v_J.$$

3.2. Proof of stability for the scheme (S1)

Denote $w_j^{n+1} = \Delta_\tau u_j^{n+1}$ $(0 \leqslant j \leqslant J)$. Making the scalar product of $w_j^{n+1}h$ $(j = 1, 2, \ldots, J-1)$ with (2.9)–(2.11), and summing up the resulting products for $j = 1, \ldots, J-1$, we get

$$\sum_{j=1}^{J-1} |w_j^{n+1}|^2 h = \sum_{j=1}^{J-1} \delta^2 u_j^{n+1} \Delta_\tau u_j^{n+1} h - \lambda w_k^{n+1}(w_{k-1}^{n+1} + w_{k+1}^{n+1})h + \lambda w_k^n (w_{k-1}^{n+1} + w_{k+1}^{n+1})h. \tag{3.3}$$

Denote $M^n = \sum_{j=0}^{J-1} |\delta u_j^n|^2 h$. Since $w_0^{n+1} = w_J^{n+1} = 0$, by Lemma 3.3, we have

$$\sum_{j=1}^{J-1} \delta^2 u_j^{n+1} \Delta_\tau u_j^{n+1} h = -\frac{1}{2\tau}(M^{n+1} - M^n) - \frac{\lambda}{2} \sum_{j=0}^{J-1} |w_{j+1}^{n+1} - w_j^{n+1}|^2 h. \tag{3.4}$$

Substitute (3.4) into (3.3) to obtain

$$\begin{aligned} \frac{1}{2\tau}(M^{n+1} - M^n) &= -\frac{\lambda h}{2}\left[\sum_{j=0}^{J-1} |w_{j+1}^{n+1} - w_j^{n+1}|^2 + \frac{2}{\lambda}\sum_{j=1}^{J-1} |w_j^{n+1}|^2 + 2w_k^{n+1}(w_{k-1}^{n+1} + w_{k+1}^{n+1}) - 2w_k^n(w_{k-1}^{n+1} + w_{k+1}^{n+1})\right] \\ &= -\frac{\lambda h}{2}\left[\sum_{j \neq k-1,k} |w_{j+1}^{n+1} - w_j^{n+1}|^2 + |w_{k-1}^{n+1} - w_k^n|^2 + |w_{k+1}^{n+1} - w_k^n|^2 + \frac{2}{\lambda}\sum_{j=1}^{J-1} |w_j^{n+1}|^2\right] \\ &\quad - \lambda h(|w_k^{n+1}|^2 - |w_k^n|^2) \leqslant -\lambda h(|w_k^{n+1}|^2 - |w_k^n|^2). \end{aligned} \tag{3.5}$$

Thus, there holds

$$M^{n+1} + 2\lambda\tau h|w_k^{n+1}|^2 \leqslant M^n + 2\lambda\tau h|w_k^n|^2, \tag{3.6}$$

and it follows:

$$M^n + 2\lambda\tau h|w_k^n|^2 \leqslant M^1 + 2\lambda\tau h|w_k^1|^2. \tag{3.7}$$

Next, consider

$$\Delta_\tau u_j^1 = \delta^2 u_j^1 = \tau\delta^2 \frac{u_j^1 - u_j^0}{\tau} + \delta^2 u_j^0 = \tau\delta^2 \Delta_\tau u_j^1 + \delta^2 u_j^0, \tag{3.8}$$

which gives, by the discrete extremum principle,

$$\|\Delta_\tau u_h^1\|_\infty \leqslant \|\delta^2 u_h^0\|_\infty. \tag{3.9}$$

Obviously,

$$M^1 \leqslant M^0. \tag{3.10}$$

Substitute (3.9) and (3.10) into (3.7) to obtain

$$M^n + 2\lambda\tau h|w_k^n| \leqslant M^0 + 2\lambda\tau h|\delta^2 u_k^0|^2.$$

Hence,

$$M^n \leqslant CM^0, \tag{3.11}$$

where C is a positive constant. So the scheme (S1) is unconditionally stable.

3.3. Proof of stability for the scheme (S2)

Making the scalar product of $w_j^{n+1}h(j = 1, 2, \dots, J-1)$ with (2.16), (2.17), and summing up the resulting products for $j = 1, \dots, J-1$, we get

$$\sum_{j=1}^{J-1} |w_j^{n+1}|^2 h = \sum_{j=1}^{J-1} \delta^2 u_j^{n+1} \Delta_\tau u_j^{n+1} h - \frac{1}{2}\lambda(w_{k+1}^{n+1} w_k^{n+1} + w_k^{n+1} w_{k-1}^{n+1}) + \frac{1}{2}\lambda(w_{k+1}^n w_k^{n+1} + w_{k-1}^n w_k^{n+1}). \tag{3.12}$$

By (3.4),

$$\begin{aligned}
\frac{1}{2\tau}(M^{n+1} - M^n) = {} & -\frac{\lambda h}{2}\left[\sum_{j\neq k-1,k} |w_{j+1}^{n+1} - w_j^{n+1}|^2 + \frac{1}{2}|w_k^{n+1} - w_{k+1}^{n+1}|^2 + \frac{1}{2}|w_{k-1}^{n+1} - w_k^{n+1}|^2 + \frac{1}{2}|w_k^{n+1} - w_{k+1}^n|^2 \right. \\
& \left. + \frac{1}{2}|w_k^{n+1} - w_{k-1}^n|^2 + \frac{2}{\lambda}\sum_{j=1}^{J-1} |w_j^{n+1}|^2\right] - \frac{1}{4}\lambda h(|w_{k+1}^{n+1}|^2 + |w_{k-1}^{n+1}|^2 - |w_{k+1}^n|^2 - |w_{k-1}^n|^2) \\
\leqslant {} & -\frac{1}{4}\lambda h(|w_{k+1}^{n+1}|^2 + |w_{k-1}^{n+1}|^2 - |w_{k+1}^n|^2 - |w_{k-1}^n|^2).
\end{aligned} \tag{3.13}$$

Thus,

$$M^{n+1} + \frac{1}{2}\lambda\tau h(|w_{k+1}^{n+1}|^2 + w_{k-1}^{n+1}|^2) \leqslant M^n + \frac{1}{2}\lambda\tau h(|w_{k+1}^n|^2 + |w_{k-1}^n|^2), \tag{3.14}$$

and it follows:

$$M^{n+1} + \frac{1}{2}\lambda\tau h(|w_{k+1}^{n+1}|^2 + w_{k-1}^{n+1}|^2) \leqslant M^1 + \frac{1}{2}\lambda\tau h(|w_{k+1}^1|^2 + |w_{k-1}^1|^2).$$

Hence,

$$M^{n+1} \leqslant CM^0.$$

So the scheme (S2) is unconditional stable. The proof of Theorem 3.1 is completed. □

3.4. Proof of convergence

At first, consider the parallel scheme (S1). $\{e_j^{n+1}\}$ satisfies the following equations:

$$\Delta_\tau e_j^{n+1} - \delta^2 e_j^{n+1} = G_j^{n+1}, \quad j = 1, 2, \dots, k-2, k, k+2, \dots, J-1, \tag{3.15}$$

$$\Delta_\tau e_{k-1}^{n+1} - \delta^2 e_{k-1}^{n+1} + \lambda\Delta_\tau e_k^{n+1} - \lambda\Delta_\tau e_k^n = G_{k-1}^{n+1}, \tag{3.16}$$

$$\Delta_\tau e_{k+1}^{n+1} - \delta^2 e_{k+1}^{n+1} + \lambda\Delta_\tau e_k^{n+1} - \lambda\Delta_\tau e_k^n = G_{k+1}^{n+1}, \tag{3.17}$$

$$e_0^{n+1} = e_J^{n+1} = 0, \quad n = 0, 1, \dots, N-1, \tag{3.18}$$

$$e_j^0 = 0, \quad j = 0, 1, \dots, J. \tag{3.19}$$

There holds $|G_j^{n+1}| \leqslant C_1(\tau + h^2)$, where C_1 is a positive constant.

Denote $v_j^{n+1} = \Delta_\tau e_j^{n+1}$. Making the scalar product of $v_j^{n+1}h$ $(j = 1, 2, \dots, J-1)$ with (3.15)–(3.17), and summing up the resulting products for $j = 1, \dots, J-1$, we get

$$\sum_{j=1}^{J-1}|v_j^{n+1}|^2h=\sum_{j=1}^{J-1}\delta^2e_j^{n+1}\Delta_\tau e_j^{n+1}h-\lambda v_k^{n+1}(v_{k-1}^{n+1}+v_{k+1}^{n+1})h+\lambda v_k^n(v_{k-1}^{n+1}+v_{k+1}^{n+1})h+\sum_{j=1}^{J-1}v_j^{n+1}G_j^{n+1}h. \tag{3.20}$$

Similar to the proof of stability for the scheme (S1), one has

$$\frac{1}{2\tau}(\|\delta e_h^{n+1}\|_2^2-\|\delta e_h^n\|_2^2)=-\frac{\lambda h}{2}\left[\sum_{j\neq k-1,k}|v_{j+1}^{n+1}-v_j^{n+1}|^2+|v_{k-1}^{n+1}-v_k^n|^2+|v_{k+1}^{n+1}-v_k^n|^2+\frac{1}{\lambda}\sum_{j=1}^{J-1}\left|\frac{1}{\sqrt{2}}G_j^{n+1}-\sqrt{2}v_j^{n+1}\right|^2\right]$$
$$+\frac{h}{4}\sum_{j=1}^{J-1}|G_j^{n+1}|^2-\lambda h(|v_k^{n+1}|^2-|v_k^n|^2)\leqslant\frac{h}{4}\sum_{j=1}^{J-1}|G_j^{n+1}|^2-\lambda h(|v_k^{n+1}|^2-|v_k^n|^2),$$

and then

$$\|\delta e_h^{n+1}\|_2^2+2\lambda\tau h|v_k^{n+1}|^2\leqslant\|\delta e_h^n\|_2^2+2\lambda\tau h|v_k^n|^2+\frac{1}{2}\tau h\sum_{j=1}^{J-1}|G_j^{n+1}|^2$$
$$\leqslant\|\delta e_h^n\|_2^2+2\lambda\tau h|v_k^n|^2+\frac{1}{2}\tau C_1^2(\tau+h^2)^2. \tag{3.21}$$

Therefore,

$$\|\delta e_h^{n+1}\|_2^2+2\lambda\tau h|v_k^{n+1}|^2\leqslant\|\delta e_h^1\|_2^2+2\lambda\tau h|v_k^1|^2+\frac{1}{2}TC_1^2(\tau+h^2)^2. \tag{3.22}$$

Notice that

$$\Delta_\tau e_j^1=\delta^2e_j^1+G_j^1.$$

Proceeding the same argument as that of (3.9) and (3.10), we have

$$|\Delta_\tau e_k^1|\leqslant|\delta^2e_k^0|+|G_j^1|\leqslant|\delta^2e_k^0|+C_1(\tau+h^2),$$
$$\|\delta e_h^1\|_2^2\leqslant\|\delta e_h^0\|_2^2+2\sum_{j=1}^{J-1}|e_j^1||G_j^1|h.$$

Since $\|e_h^1\|_2\leqslant C\|\delta e_h^1\|_2$, and $e_j^0=0$, we obtain

$$|\Delta_\tau e_k^1|\leqslant C_1(\tau+h^2), \tag{3.23}$$
$$\|\delta e_h^1\|_2^2\leqslant C_2(\tau+h^2)^2. \tag{3.24}$$

Substitute (3.23) and (3.24) into (3.22) to obtain

$$\|\delta e_h^{n+1}\|_2^2\leqslant C(\tau+h^2)^2. \tag{3.25}$$

So the scheme (S1) is unconditional convergent and the convergence order is second order. The similar result holds for the scheme (S2). The proof of Theorem 3.2 is finished. □

It is straightforward to extend the results on two subdomains to allow for multi-sub-domains.

4. Parallel schemes for two-dimensional problem

4.1. Problem and notation

In this section, we will construct the parallel difference scheme for the following two-dimensional parabolic problem:

$$U_t=U_{xx}+U_{yy},\quad (x,y,t)\in\Omega\times(0,T], \tag{4.1}$$
$$U(x,y,t)=0,\quad (x,y,t)\in\partial\Omega\times(0,T], \tag{4.2}$$
$$U(x,y,0)=U_0(x,y),\quad (x,y)\in\Omega, \tag{4.3}$$

where $\Omega=\{(x,y)|0<x<1,0<y<1\}$, $U_0(x,y)$ is a initial value function. The initial and boundary data satisfy the compatibility condition that $U_0(x,y)=0$ for $(x,y)\in\partial\Omega$.

Divide the domain $\Omega\times(0,T]$ by $x_i=ih$, $y_j=jh$, $t^n=n\tau$. Let $\tau=T/N$, $h=1/J$, $\lambda=\tau/h^2$. For a function $\phi(x,y,t)$ defined at mesh point (x_i,y_j,t^n), let $\phi^n_{i,j}=\phi(x_i,y_j,t^n)$. Define the difference operators:

$$\delta_1u^n_{i,j}=\frac{1}{h}(u^n_{i+1,j}-u^n_{i,j}),\quad \delta_1^2u^n_{i,j}=\frac{1}{h}(\delta_1u^n_{i,j}-\delta_1u^n_{i-1,j}),$$
$$\delta_2u^n_{i,j}=\frac{1}{h}(u^n_{i,j+1}-u^n_{i,j}),\quad \delta_2^2u^n_{i,j}=\frac{1}{h}(\delta_2u^n_{i,j}-\delta_2u^n_{i,j-1}),\quad \Delta_\tau u^{n+1}_{i,j}=\frac{1}{\tau}(u^{n+1}_{i,j}-u^n_{i,j}).$$

Define the discrete norm as follows:

$$\|u^n_h\|_2^2=\sum_{i,j=0}^{J}|u^n_{i,j}|^2h^2,\quad \|\delta u^n_h\|_2^2=\sum_{i=0}^{J-1}\sum_{j=0}^{J}|\delta_1u^n_{i,j}|^2h^2+\sum_{i=0}^{J}\sum_{j=0}^{J-1}|\delta_2u^n_{i,j}|^2h^2.$$

Next, we will extend the schemes (S1) and (S2) to two-dimensional problem (4.1)–(4.3).

4.2. Parallel scheme 3 (S3)

Let k and l be a positive integer such that $2<k<J-2$ and $2<l<J-2$, respectively. Assume $\{u^{n-1}_{i,j}\}$ and $\{u^n_{i,j}\}$ are given. Firstly compute the values of $\{u^{n+1}_{i,j}|i\neq k,j\neq l\}$ by the following equations.

$$\Delta_\tau u^{n+1}_{i,j}=\delta_1^2u^{n+1}_{i,j}+\delta_2^2u^{n+1}_{i,j},\quad i\neq k-1,k,k+1;j\neq l-1,l,l+1, \tag{4.4}$$

$$\Delta_\tau u^{n+1}_{k-1,j}=\frac{1}{h^2}((2u^n_{k,j}-u^{n-1}_{k,j})-2u^{n+1}_{k-1,j}+u^{n+1}_{k-2,j})+\delta_2^2u^{n+1}_{k-1,j},\quad j\neq l-1,l,l+1, \tag{4.5}$$
$$\Delta_\tau u^{n+1}_{k+1,j}=\frac{1}{h^2}(u^{n+1}_{k+2,j}-2u^{n+1}_{k+1,j}+(2u^n_{k,j}-u^{n-1}_{k,j}))+\delta_2^2u^{n+1}_{k+1,j},\quad j\neq l-1,l,l+1,$$
$$\Delta_\tau u^{n+1}_{i,l-1}=\frac{1}{h^2}((2u^n_{i,l}-u^{n-1}_{i,l})-2u^{n+1}_{i,l-1}+u^{n+1}_{i,l-2})+\delta_1^2u^{n+1}_{i,l-1},\quad i\neq k-1,k,k+1,$$
$$\Delta_\tau u^{n+1}_{i,l+1}=\frac{1}{h^2}(u^{n+1}_{i,l+2}-2u^{n+1}_{i,l+1}+(2u^n_{i,l}-u^{n-1}_{i,l}))+\delta_1^2u^{n+1}_{i,l+1},\quad i\neq k-1,k,k+1,$$
$$\Delta_\tau u^{n+1}_{k-1,l-1}=\frac{1}{h^2}((2u^n_{k,l-1}-u^{n-1}_{k,l-1})-2u^{n+1}_{k-1,l-1}+u^{n+1}_{k-2,l-1})+\frac{1}{h^2}((2u^n_{k-1,l}-u^{n-1}_{k-1,l})-2u^{n+1}_{k-1,l-1}+u^{n+1}_{k-1,l-2}),$$
$$\Delta_\tau u^{n+1}_{k+1,l-1}=\frac{1}{h^2}(u^{n+1}_{k+2,l-1}-2u^{n+1}_{k+1,l-1}+(2u^n_{k,l-1}-u^{n-1}_{k,l-1}))+\frac{1}{h^2}((2u^n_{k+1,l}-u^{n-1}_{k+1,l})-2u^{n+1}_{k+1,l-1}+u^{n+1}_{k+1,l-2}),$$
$$\Delta_\tau u^{n+1}_{k-1,l+1}=\frac{1}{h^2}((2u^n_{k,l+1}-u^{n-1}_{k,l+1})-2u^{n+1}_{k-1,l+1}+u^{n+1}_{k-2,l+1})+\frac{1}{h^2}(u^{n+1}_{k-1,l+2}-2u^{n+1}_{k-1,l+1}+(2u^n_{k-1,l}-u^{n-1}_{k-1,l})),$$
$$\Delta_\tau u^{n+1}_{k+1,l+1}=\frac{1}{h^2}(u^{n+1}_{k+2,l+1}-2u^{n+1}_{k+1,l+1}+(2u^n_{k,l+1}-u^{n-1}_{k,l+1}))+\frac{1}{h^2}(u^{n+1}_{k+1,l+2}-2u^{n+1}_{k+1,l+1}+(2u^n_{k+1,l}-u^{n-1}_{k+1,l})). \tag{4.6}$$

Eq. (4.4) is fully implicit. Eqs. (4.5)–(4.6) are obtained by replacing $u^{n+1}_{i,j}$ with $2u^n_{i,j}-u^{n-1}_{i,j}$ ($i=k$ or $j=l$) in a stencil of implicit scheme, respectively. Therefore, the global problem is fully decoupled and can thus be computed in parallel. Then compute the interface values except the cross point $(i,j)=(k,l)$ by the following equations:

$$\Delta_\tau u^{n+1}_{k,j}=\delta_1^2u^{n+1}_{k,j}+\delta_2^2u^{n+1}_{k,j},\quad j\neq l-1,l,l+1, \tag{4.7}$$
$$\Delta_\tau u^{n+1}_{i,l}=\delta_1^2u^{n+1}_{i,l}+\delta_2^2u^{n+1}_{i,l},\quad i\neq k-1,k,k+1, \tag{4.8}$$
$$\Delta_\tau u^{n+1}_{k,l-1}=\frac{1}{h^2}((2u^n_{k,l}-u^{n-1}_{k,l})-2u^{n+1}_{k,l-1}+u^{n+1}_{k,l-2})+\delta_1^2u^{n+1}_{k,l-1}, \tag{4.9}$$
$$\Delta_\tau u^{n+1}_{k,l+1}=\frac{1}{h^2}(u^{n+1}_{k,l+2}-2u^{n+1}_{k,l+1}+(2u^n_{k,l}-u^{n-1}_{k,l}))+\delta_1^2u^{n+1}_{k,l+1},$$
$$\Delta_\tau u^{n+1}_{k-1,l}=\frac{1}{h^2}((2u^n_{k,l}-u^{n-1}_{k,l})-2u^{n+1}_{k-1,l}+u^{n+1}_{k-2,l})+\delta_2^2u^{n+1}_{k-1,l},$$
$$\Delta_\tau u^{n+1}_{k+1,l}=\frac{1}{h^2}(u^{n+1}_{k+2,l}-2u^{n+1}_{k+1,l}+(2u^n_{k,l}-u^{n-1}_{k,l}))+\delta_2^2u^{n+1}_{k+1,l}. \tag{4.10}$$

Eqs. (4.9)–(4.10) are formulated by replacing $u_{k,l}^{n+1}$ with $2u_{k,l}^{n} - u_{k,l}^{n-1}$ in a stencil of implicit scheme, respectively. Therefore, the interface values except the cross point $(i,j) = (k,l)$ can be computed in parallel. And then finally, compute the value of cross point $(i,j) = (k,l)$ by the fully implicit scheme:

$$\Delta_\tau u_{k,l}^{n+1} = \delta_1^2 u_{k,l}^{n+1} + \delta_2^2 u_{k,l}^{n+1}. \tag{4.11}$$

It is obvious that the truncation error of the scheme (4.4)–(4.11) is $O(\tau + h^2)$. For $n = 0$, use implicit scheme

$$\Delta_\tau u_{i,j}^{1} = \delta_1^2 u_{i,j}^{1} + \delta_2^2 u_{i,j}^{1}. \tag{4.12}$$

Eqs. (4.4)–(4.11) are equivalent to the following equations:

$$\Delta_\tau u_{i,j}^{n+1} = \delta_1^2 u_{i,j}^{n+1} + \delta_2^2 u_{i,j}^{n+1}, \quad i \neq k-1, k+1; j \neq l-1, l+1, \tag{4.13}$$

$$\begin{aligned}
\Delta_\tau u_{i,j}^{n+1} &= \delta_1^2 u_{i,j}^{n+1} + \delta_2^2 u_{i,j}^{n+1} - \lambda\Delta_\tau u_{k,j}^{n+1} + \lambda\Delta_\tau u_{k,j}^{n}, \quad i = k-1, k+1; j \neq l-1, l+1,\\
\Delta_\tau u_{i,j}^{n+1} &= \delta_1^2 u_{i,j}^{n+1} + \delta_2^2 u_{i,j}^{n+1} - \lambda\Delta_\tau u_{i,l}^{n+1} + \lambda\Delta_\tau u_{i,l}^{n}, \quad j = l-1, l+1; i \neq k-1, k+1,\\
\Delta_\tau u_{k-1,l-1}^{n+1} &= \delta_1^2 u_{k-1,l-1}^{n+1} + \delta_2^2 u_{k-1,l-1}^{n+1} - \lambda\Delta_\tau u_{k,l-1}^{n+1} + \lambda\Delta_\tau u_{k,l-1}^{n} - \lambda\Delta_\tau u_{k-1,l}^{n+1} + \lambda\Delta_\tau u_{k-1,l}^{n},\\
\Delta_\tau u_{k+1,l-1}^{n+1} &= \delta_1^2 u_{k+1,l-1}^{n+1} + \delta_2^2 u_{k+1,l-1}^{n+1} - \lambda\Delta_\tau u_{k,l-1}^{n+1} + \lambda\Delta_\tau u_{k,l-1}^{n} - \lambda\Delta_\tau u_{k+1,l}^{n+1} + \lambda\Delta_\tau u_{k+1,l}^{n},\\
\Delta_\tau u_{k-1,l+1}^{n+1} &= \delta_1^2 u_{k-1,l+1}^{n+1} + \delta_2^2 u_{k-1,l+1}^{n+1} - \lambda\Delta_\tau u_{k,l+1}^{n+1} + \lambda\Delta_\tau u_{k,l+1}^{n} - \lambda\Delta_\tau u_{k-1,l}^{n+1} + \lambda\Delta_\tau u_{k-1,l}^{n},\\
\Delta_\tau u_{k+1,l+1}^{n+1} &= \delta_1^2 u_{k+1,l+1}^{n+1} + \delta_2^2 u_{k+1,l+1}^{n+1} - \lambda\Delta_\tau u_{k,l+1}^{n+1} + \lambda\Delta_\tau u_{k,l+1}^{n} - \lambda\Delta_\tau u_{k+1,l}^{n+1} + \lambda\Delta_\tau u_{k+1,l}^{n}.
\end{aligned} \tag{4.14}$$

4.3. Parallel scheme 4 (S4)

Let k and l be positive integers such that $2 < k < J-2$ and $2 < l < J-2$, respectively. Assume $\{u_{i,j}^{n-1}\}$ and $\{u_{i,j}^{n}\}$ are given. Firstly, compute the values of $\{u_{i,j}^{n+1} | i \neq k, j \neq l\}$ by the following equations.

$$\Delta_\tau u_{i,j}^{n+1} = \delta_1^2 u_{i,j}^{n+1} + \delta_2^2 u_{i,j}^{n+1}, \quad i \neq k; j \neq l, \tag{4.15}$$

$$\Delta_\tau \bar{u}_{k,j}^{n+1} = \frac{1}{h^2}((2u_{k+1,j}^{n} - u_{k+1,j}^{n-1}) - \bar{u}_{k,j}^{n+1} - (\bar{u}_{k,j}^{n+1} - u_{k-1,j}^{n+1})) + \delta_2^2 \bar{u}_{k,j}^{n+1}, \quad j \neq l, \tag{4.16}$$

$$\begin{aligned}
\Delta_\tau \tilde{u}_{k,j}^{n+1} &= \frac{1}{h^2}(u_{k+1,j}^{n+1} - \tilde{u}_{k,j}^{n+1} - (\tilde{u}_{k,j}^{n+1} - (2u_{k-1,j}^{n} - u_{k-1,j}^{n-1}))) + \delta_2^2 \tilde{u}_{k,j}^{n+1}, \quad j \neq l,\\
\Delta_\tau \bar{u}_{i,l}^{n+1} &= \frac{1}{h^2}((2u_{i,l+1}^{n} - u_{i,l+1}^{n-1}) - \bar{u}_{i,l}^{n+1} - (\bar{u}_{i,l}^{n+1} - u_{i,l-1}^{n+1})) + \delta_1^2 \bar{u}_{i,l}^{n+1}, \quad i \neq k,\\
\Delta_\tau \tilde{u}_{i,l}^{n+1} &= \frac{1}{h^2}(u_{i,l+1}^{n+1} - \tilde{u}_{i,l}^{n+1} - (\tilde{u}_{i,l}^{n+1} - (2u_{i,l-1}^{n} - u_{i,l-1}^{n-1}))) + \delta_1^2 \tilde{u}_{i,l}^{n+1}, \quad i \neq k,
\end{aligned} \tag{4.17}$$

$$\Delta_\tau \breve{u}_{k,l}^{n+1} = \frac{1}{h^2}((2u_{k+1,l}^{n} - u_{k+1,l}^{n-1}) - \breve{u}_{k,l}^{n+1} - (\breve{u}_{k,l}^{n+1} - u_{k-1,l}^{n+1})) + \frac{1}{h^2}((2u_{k,l+1}^{n} - u_{k,l+1}^{n-1}) - \breve{u}_{k,l}^{n+1} - (\breve{u}_{k,l}^{n+1} - u_{k,l-1}^{n+1})), \tag{4.18}$$

$$\begin{aligned}
\Delta_\tau \hat{u}_{k,l}^{n+1} &= \frac{1}{h^2}(u_{k+1,l}^{n+1} - \hat{u}_{k,l}^{n+1} - (\hat{u}_{k,l}^{n+1} - (2u_{k-1,l}^{n} - u_{k-1,l}^{n-1}))) + \frac{1}{h^2}((2u_{k,l+1}^{n} - u_{k,l+1}^{n-1}) - \hat{u}_{k,l}^{n+1} - (\hat{u}_{k,l}^{n+1} - u_{k,l-1}^{n+1})),\\
\Delta_\tau \check{u}_{k,l}^{n+1} &= \frac{1}{h^2}((2u_{k+1,l}^{n} - u_{k+1,l}^{n-1}) - \check{u}_{k,l}^{n+1} - (\check{u}_{k,l}^{n+1} - u_{k-1,l}^{n+1})) + \frac{1}{h^2}(u_{k,l+1}^{n+1} - \check{u}_{k,l}^{n+1} - (\check{u}_{k,l}^{n+1} - (2u_{k,l-1}^{n} - u_{k,l-1}^{n-1}))),\\
\Delta_\tau \acute{u}_{k,l}^{n+1} &= \frac{1}{h^2}(u_{k+1,l}^{n+1} - \acute{u}_{k,l}^{n+1} - (\acute{u}_{k,l}^{n+1} - (2u_{k-1,l}^{n} - u_{k-1,l}^{n-1}))) + \frac{1}{h^2}(u_{k,l+1}^{n+1} - \acute{u}_{k,l}^{n+1} - (\acute{u}_{k,l}^{n+1} - (2u_{k,l-1}^{n} - u_{k,l-1}^{n-1}))).
\end{aligned} \tag{4.19}$$

There are different values at mesh points $x = x_k$ or $y = y_l$ in different processors. Similar to the parallel scheme (S2), take the average of two values as the value at mesh points $(x = x_k, y \neq y_l)$ and $(x \neq x_k, y = y_l)$, respectively, and take the average of the four values as the value at mesh point $(x = x_k, y = y_l)$, i.e.

$$u_{k,j}^{n+1} = \frac{1}{2}(\bar{u}_{k,j}^{n+1} + \tilde{u}_{k,j}^{n+1}), \quad j \neq l, \tag{4.20}$$

$$u_{i,l}^{n+1} = \frac{1}{2}(\bar{u}_{i,l}^{n+1} + \tilde{u}_{i,l}^{n+1}), \quad i \neq k, \tag{4.21}$$

$$u_{k,l}^{n+1} = \frac{1}{4}(\hat{u}_{k,l}^{n+1} + \breve{u}_{k,l}^{n+1} + \check{u}_{k,l}^{n+1} + \acute{u}_{k,l}^{n+1}). \tag{4.22}$$

In Eqs. (4.15)–(4.22), (4.15) is fully implicit, and (4.16)–(4.17) are obtained by replacing $u^{n+1}_{k+1,j}$ with $2u^{n}_{k+1,j}-u^{n-1}_{k+1,j}$, $u^{n+1}_{k-1,j}$ with $2u^{n}_{k-1,j}-u^{n-1}_{k-1,j}$, $u^{n+1}_{i,l+1}$ with $2u^{n}_{i,l+1}-u^{n-1}_{i,l+1}$, $u^{n+1}_{i,l-1}$ with $2u^{n}_{i,l-1}-u^{n-1}_{i,l-1}$ in the fully implicit stencil, respectively; And similarly, (4.18)–(4.19) are obtained by replacing $u^{n+1}_{k+1,l}$, $u^{n+1}_{k,l+1}$ with $2u^{n}_{k+1,l}-u^{n-1}_{k+1,l}$, $2u^{n}_{k,l+1}-u^{n-1}_{k,l+1}$, $u^{n+1}_{k-1,l}$, $u^{n+1}_{k,l+1}$ with $2u^{n}_{k-1,l}-u^{n-1}_{k-1,l}$, $2u^{n}_{k,l+1}-u^{n-1}_{k,l+1}$, $u^{n+1}_{k+1,l}$, $u^{n+1}_{k,l-1}$ with $2u^{n}_{k+1,l}-u^{n-1}_{k+1,l}$, $2u^{n}_{k,l-1}-u^{n-1}_{k,l-1}$, $u^{n+1}_{k-1,l}$, $u^{n+1}_{k,l-1}$ with $2u^{n}_{k-1,l}-u^{n-1}_{k-1,l}$, $2u^{n}_{k,l-1}-u^{n-1}_{k,l-1}$ in the fully implicit stencil, respectively. Therefore the global problem is fully decoupled and the interior values in sub-domain can thus be computed in parallel. The interface values can also be computed in parallel. It is obvious that the truncation error of the scheme is $O(\tau+h^2)$.

Eqs. (4.15)–(4.19) are equivalent to the following equations.

$$\Delta_\tau u^{n+1}_{i,j}=\delta_1^2u^{n+1}_{i,j}+\delta_2^2u^{n+1}_{i,j},\quad i\neq k;j\neq l, \tag{4.23}$$

$$\Delta_\tau u^{n+1}_{k,j}=\delta_1^2u^{n+1}_{k,j}+\delta_2^2u^{n+1}_{k,j}-\frac{1}{2}\lambda(\Delta_\tau u^{n+1}_{k+1,j}+\Delta_\tau u^{n+1}_{k-1,j})+\frac{1}{2}(\lambda\Delta_\tau u^{n}_{k+1,j}+\Delta_\tau u^{n}_{k-1,j}),\quad j\neq l,$$

$$\Delta_\tau u^{n+1}_{i,l}=\delta_1^2u^{n+1}_{i,l}+\delta_2^2u^{n+1}_{i,l}-\frac{1}{2}\lambda(\Delta_\tau u^{n+1}_{i,l+1}+\Delta_\tau u^{n+1}_{i,l-1})+\frac{1}{2}\lambda(\Delta_\tau u^{n}_{i,l+1}+\Delta_\tau u^{n}_{i,l-1}),\quad i\neq k, \tag{4.24}$$

$$\Delta_\tau u^{n+1}_{k,l}=\delta_1^2u^{n+1}_{k,l}+\delta_2^2u^{n+1}_{k,l}-\frac{1}{2}\lambda(\Delta_\tau u^{n+1}_{k+1,l}+\Delta_\tau u^{n+1}_{k-1,l}+\Delta_\tau u^{n+1}_{k,l+1}+\Delta_\tau u^{n+1}_{k,l-1})$$
$$+\frac{1}{2}\lambda(\Delta_\tau u^{n}_{k+1,l}+\Delta_\tau u^{n}_{k-1,l}+\Delta_\tau u^{n}_{k,l+1}+\Delta_\tau u^{n}_{k,l-1}). \tag{4.25}$$

5. Unconditional stability and convergence of the schemes (S3) and (S4)

5.1. Theorem of stability and convergence

Similar to one-dimensional problem, there are the following theorems of stability and convergence.

Theorem 5.1. *Parallel difference schemes (S3) and (S4) are unconditional stable in the sense of discrete H^1 norm, i.e., for any given λ,*

$$\|\delta u^n_h\|_2\leqslant C\|\delta u^0_h\|_2^2,\quad \forall n=1,2,\dots,N. \tag{5.1}$$

where C is a positive constant.

Let $e^n_{i,j}=U^n_{i,j}-u^n_{i,j}$. The approximate solution satisfies the following a priori error estimate:

Theorem 5.2. *Parallel schemes (S3) and (S4) are unconditional convergent, i.e., for any given $\lambda>0$,*

$$\|\delta e^n_h\|_2\leqslant C(\tau+h^2),\quad \forall n=1,2,\dots,N. \tag{5.2}$$

where C is a positive constant.

5.2. Proof of stability for S3

Let $w^{n+1}_{i,j}=\Delta_\tau u^{n+1}_{i,j}$. Making the scalar product of $w^{n+1}_{i,j}h^2$ with (4.13)–(4.14), and summing up the resulting products for i,j, we get

$$\sum_{i,j=1}^{J-1}|w^{n+1}_{i,j}|^2h^2=\sum_{i,j=1}^{J-1}(\delta_1^2u^{n+1}_{i,j}+\delta_2^2u^{n+1}_{i,j})\Delta_\tau u^{n+1}_{i,j}h^2-\lambda\sum_{\substack{j=1\\ i=k-1,k+1}}^{J-1}w^{n+1}_{k,j}w^{n+1}_{i,j}h^2+\lambda\sum_{\substack{j=1\\ i=k-1,k+1}}^{J-1}w^{n}_{k,j}w^{n+1}_{i,j}h^2$$
$$-\lambda\sum_{\substack{i=1\\ j=l-1,l+1}}^{J-1}w^{n+1}_{i,l}w^{n+1}_{i,j}h^2+\lambda\sum_{\substack{i=1\\ j=l-1,l+1}}^{J-1}w^{n}_{i,l}w^{n+1}_{i,j}h^2. \tag{5.3}$$

Let $M_1^n = \sum_{i=0,j=1}^{J-1} |\delta_1 u_{i,j}^n|^2 h^2$, $M_2^n = \sum_{i=1,j=0}^{J-1} |\delta_2 u_{i,j}^n|^2 h^2$. By the Lemma 3.3,

$$\sum_{i,j=1}^{J-1} \delta_1^2 u_{i,j}^{n+1} \Delta_\tau u_{i,j}^{n+1} h^2 = -\frac{1}{2\tau}(M_1^{n+1} - M_1^n) - \frac{\lambda}{2} \sum_{\substack{i=0 \\ j=1}}^{J-1} |w_{i+1,j}^{n+1} - w_{i,j}^{n+1}|^2 h^2, \tag{5.4}$$

$$\sum_{i,j=1}^{J-1} \delta_2^2 u_{i,j}^{n+1} \Delta_\tau u_{i,j}^{n+1} h^2 = -\frac{1}{2\tau}(M_2^{n+1} - M_2^n) - \frac{\lambda}{2} \sum_{\substack{i=1 \\ j=0}}^{J-1} |w_{i+1,j}^{n+1} - w_{i,j}^{n+1}|^2 h^2. \tag{5.5}$$

Let $M^n = M_1^n + M_2^n$, and substitute (5.4) and (5.5) into (5.3) to find

$$\begin{aligned}
\frac{1}{2\tau}(M^{n+1} - M^n) = &-\frac{\lambda h^2}{2}\left(\sum_{\substack{j=1 \\ i\neq k-1,k}}^{J-1} |w_{i+1,j}^{n+1} - w_{i,j}^{n+1}|^2 + \sum_{\substack{i=1 \\ j\neq l-1,l}}^{J-1} |w_{i,j+1}^{n+1} - w_{i,j}^{n+1}|^2 + \frac{2}{\lambda}\sum_{i,j=1}^{J-1} |w_{i,j}^{n+1}|^2 \right. \\
&\left. + \sum_{j=1}^{J-1}(|w_{k,j}^n - w_{k-1,j}^{n+1}|^2 + |w_{k,j}^n - w_{k+1,j}^{n+1}|^2) + \sum_{i=1}^{J-1}(|w_{i,l}^n - w_{i,l-1}^{n+1}|^2 + |w_{i,l}^n - w_{i,l+1}^{n+1}|^2) \right) \\
&- \lambda h^2 \sum_{j=1}^{J-1}(|w_{k,j}^{n+1}|^2 - |w_{k,j}^n|^2) - \lambda h^2 \sum_{i=1}^{J-1}(|w_{i,l}^{n+1}|^2 - |w_{i,l}^n|^2) \leqslant -\lambda h^2 \sum_{j=1}^{J-1}(|w_{k,j}^{n+1}|^2 - |w_{k,j}^n|^2) \\
&- \lambda h^2 \sum_{i=1}^{J-1}(|w_{i,l}^{n+1}|^2 - |w_{i,l}^n|^2),
\end{aligned} \tag{5.6}$$

and it follows:

$$M^{n+1} + 2\lambda\tau h^2 \left(\sum_{j=1}^{J-1} |w_{k,j}^{n+1}|^2 + \sum_{i=1}^{J-1} |w_{i,l}^{n+1}|^2 \right) \leqslant M^n + 2\lambda\tau h^2 \left(\sum_{j=1}^{J-1} |w_{k,j}^n|^2 + \sum_{i=1}^{J-1} |w_{i,l}^n|^2 \right). \tag{5.7}$$

Hence,

$$M^{n+1} + 2\lambda\tau h^2 \left(\sum_{j=1}^{J-1} |w_{k,j}^{n+1}|^2 + \sum_{i=1}^{J-1} |w_{i,l}^{n+1}|^2 \right) \leqslant M^1 + 2\lambda\tau h^2 \left(\sum_{j=1}^{J-1} |w_{k,j}^1|^2 + \sum_{i=1}^{J-1} |w_{i,l}^1|^2 \right). \tag{5.8}$$

Similar to one-dimensional problem one finds

$$M^{n+1} \leqslant CM^0, \tag{5.9}$$

where C is a positive constant.

5.3. Proof of stability for S4

Making the scalar product of $w_{i,j}^{n+1} h^2$ with (4.23)–(4.25), and summing up the resulting products for i, j, we get

$$\begin{aligned}
\sum_{i,j=1}^{J-1} |w_{i,j}^{n+1}|^2 h^2 = &\sum_{i,j=1}^{J-1} (\delta_1^2 u_{i,j}^{n+1} + \delta_2^2 u_{i,j}^{n+1}) \Delta_\tau u_{i,j}^{n+1} h^2 - \frac{1}{2}\lambda \sum_{j=1}^{J-1} (w_{k+1,j}^{n+1} + w_{k-1,j}^{n+1}) w_{k,j}^{n+1} h^2 \\
&+ \frac{1}{2}\lambda \sum_{j=1}^{J-1} (w_{k+1,j}^n + w_{k-1,j}^{n+1}) w_{k,j}^{n+1} h^2 - \frac{1}{2}\lambda \sum_{i=1}^{J-1} (w_{i,l+1}^{n+1} + w_{i,l-1}^{n+1}) w_{i,l}^{n+1} h^2 \\
&+ \frac{1}{2}\lambda \sum_{i=1}^{J-1} (w_{i,l+1}^n + w_{i,l-1}^{n+1}) w_{i,l}^{n+1} h^2.
\end{aligned}$$

By (5.4) and (5.5),

$$\begin{aligned}\frac{1}{2\tau}(M^{n+1}-M^n) = &-\frac{\lambda h^2}{2}\left(\sum_{\substack{j=1\\ i\neq k-1,k}}^{J-1}|w_{i+1,j}^{n+1}-w_{i,j}^{n+1}|^2+\sum_{\substack{i=1\\ j\neq l-1,l}}^{J-1}|w_{i,j+1}^{n+1}-w_{i,j}^{n+1}|^2+\frac{1}{2}\sum_{j=1}^{J-1}(|w_{k,j}^{n+1}-w_{k-1,j}^{n+1}|^2\right.\\ &+|w_{k+1,j}^{n+1}-w_{k,j}^{n+1}|^2+|w_{k+1,j}^{n}-w_{k,j}^{n+1}|^2+|w_{k-1,j}^{n}-w_{k,j}^{n+1}|^2+|w_{k-1,j}^{n+1}|^2+|w_{k+1,j}^{n+1}|^2\\ &-|w_{k-1,j}^{n}|^2-|w_{k+1,j}^{n}|^2)+\frac{1}{2}\sum_{i=1}^{J-1}(|w_{i,l}^{n+1}-w_{i,l-1}^{n+1}|^2+|w_{i,l+1}^{n+1}-w_{i,l}^{n+1}|^2+|w_{i,l+1}^{n}-w_{i,l}^{n+1}|^2\\ &\left.+|w_{i,l-1}^{n}-w_{i,l}^{n+1}|^2+|w_{i,l-1}^{n+1}|^2+|w_{i,l+1}^{n+1}|^2-|w_{i,l-1}^{n}|^2-|w_{i,l+1}^{n}|^2)\right)\\ \leqslant &-\frac{\lambda h^2}{4}\left[\sum_{j=1}^{J-1}(|w_{k-1,j}^{n+1}|^2+|w_{k+1,j}^{n+1}|^2-|w_{j-1,j}^{n}|^2-|w_{k+1,j}^{n}|^2)\right.\\ &\left.+\sum_{i=1}^{J-1}(|w_{i,l-1}^{n+1}|^2+|w_{i,l+1}^{n+1}|^2-|w_{i,l-1}^{n}|^2-|w_{i,l+1}^{n}|^2)\right],\end{aligned}\tag{5.10}$$

and it follows:

$$\begin{aligned}&M^{n+1}+\frac{\lambda\tau h^2}{2}\left[\sum_{j=1}^{J-1}(|w_{k-1,j}^{n+1}|^2+w_{k+1,j}^{n+1}|^2)+\sum_{i=1}^{J-1}(|w_{i,l-1}^{n+1}|^2+|w_{i,l+1}^{n+1}|^2)\right]\\ &\quad\leqslant M^n+\frac{\lambda\tau h^2}{2}\left[\sum_{j=1}^{J-1}(|w_{k-1,j}^{n}|^2+|w_{k+1,j}^{n}|^2)+\sum_{i=1}^{J-1}(|w_{i,l-1}^{n}|^2+|w_{i,l+1}^{n}|^2)\right].\end{aligned}\tag{5.11}$$

Thus,

$$M^{n+1}\leqslant CM^0.\tag{5.12}$$

5.4. Proof of convergence for S3

Consider the parallel scheme S3. $\{e_{i,j}^{n+1}\}$ satisfy

$$\Delta_\tau e_{i,j}^{n+1}=\delta_1^2e_{i,j}^{n+1}+\delta_2^2e_{i,j}^{n+1}+G_{i,j}^{n+1},\quad i\neq k-1,k+1;j\neq l-1,l+1,\tag{5.13}$$

$$\begin{aligned}&\Delta_\tau e_{i,j}^{n+1}=\delta_1^2e_{i,j}^{n+1}+\delta_2^2e_{i,j}^{n+1}-\lambda\Delta_\tau e_{k,j}^{n+1}+\lambda\Delta_\tau e_{k,j}^{n}+G_{i,j}^{n+1},\quad i=k-1,k+1;j\neq l-1,l,l+1,\\ &\Delta_\tau e_{i,j}^{n+1}=\delta_1^2e_{i,j}^{n+1}+\delta_2^2e_{i,j}^{n+1}-\lambda\Delta_\tau e_{i,l}^{n+1}+\lambda\Delta_\tau e_{i,l}^{n}+G_{i,j}^{n+1},\quad j=l-1,l+1;i\neq k-1,k,k+1,\\ &\Delta_\tau e_{k-1,l-1}^{n+1}=\delta_1^2e_{k-1,l-1}^{n+1}+\delta_2^2e_{k-1,l-1}^{n+1}-\lambda\Delta_\tau e_{k,l-1}^{n+1}+\lambda\Delta_\tau e_{k,l-1}^{n}-\lambda\Delta_\tau e_{k-1,l}^{n+1}+\lambda\Delta_\tau e_{k-1,l}^{n}+G_{k-1,l-1}^{n+1},\\ &\Delta_\tau e_{k+1,l-1}^{n+1}=\delta_1^2e_{k+1,l-1}^{n+1}+\delta_2^2e_{k+1,l-1}^{n+1}-\lambda\Delta_\tau e_{k,l-1}^{n+1}+\lambda\Delta_\tau e_{k,l-1}^{n}-\lambda\Delta_\tau e_{k+1,l}^{n+1}+\lambda\Delta_\tau e_{k+1,l}^{n}+G_{k+1,l-1}^{n+1},\\ &\Delta_\tau e_{k-1,l+1}^{n+1}=\delta_1^2e_{k-1,l+1}^{n+1}+\delta_2^2e_{k-1,l+1}^{n+1}-\lambda\Delta_\tau e_{k,l+1}^{n+1}+\lambda\Delta_\tau e_{k,l+1}^{n}-\lambda\Delta_\tau e_{k-1,l}^{n+1}+\lambda\Delta_\tau e_{k-1,l}^{n}+G_{k-1,l+1}^{n+1},\\ &\Delta_\tau e_{k+1,l+1}^{n+1}=\delta_1^2e_{k+1,l+1}^{n+1}+\delta_2^2e_{k+1,l+1}^{n+1}-\lambda\Delta_\tau e_{k,l+1}^{n+1}+\lambda\Delta_\tau e_{k,l+1}^{n}-\lambda\Delta_\tau e_{k+1,l}^{n+1}+\lambda\Delta_\tau e_{k+1,l}^{n}+G_{k+1,l+1}^{n+1},\end{aligned}\tag{5.14}$$

$$e_{i,0}^{n+1}=e_{i,J}^{n+1}=e_{0,j}^{n+1}=e_{J,j}^{n+1}=0,\quad i,j=1,\dots,J,\tag{5.15}$$

$$e_{i,j}^0=0,\quad i,j=1,\dots,J.\tag{5.16}$$

There holds $|G_{i,j}^{n+1}|\leqslant C_3(\tau+h^2)$, and C_3 is a positive constant.

Denote $v_{i,j}^{n+1}=\Delta_\tau e_{i,j}^{n+1}$. Making the scalar product of $v_{i,j}^{n+1}h^2$ with (5.13)–(5.14), and summing up the resulting products for i,j, we get

$$\sum_{i,j=1}^{J-1}|v_{i,j}^{n+1}|^2h^2=\sum_{i,j=1}^{J-1}(\delta_1^2e_{i,j}^{n+1}+\delta_2^2e_{i,j}^{n+1})\Delta_\tau e_{i,j}^{n+1}h^2-\lambda\sum_{\substack{j=1\\i=k-1,k+1}}^{J-1}v_{k,j}^{n+1}v_{i,j}^{n+1}h^2+\lambda\sum_{\substack{j=1\\i=k-1,k+1}}^{J-1}v_{k,j}^{n}v_{i,j}^{n+1}h^2$$
$$-\lambda\sum_{\substack{i=1\\j=l-1,l+1}}^{J-1}v_{i,l}^{n+1}v_{i,j}^{n+1}h^2+\lambda\sum_{\substack{i=1\\j=l-1,l+1}}^{J-1}v_{i,l}^{n}v_{i,j}^{n+1}h^2+\sum_{i,j=1}^{J-1}v_{i,j}^{n+1}G_{i,j}^{n+1}h^2. \tag{5.17}$$

Similar to the proof of stability for the scheme (S3), it gives

$$\begin{aligned}\frac{1}{2\tau}(\|e_h^{n+1}\|_2^2-\|e_h^n\|_2^2)=&-\frac{\lambda h^2}{2}\Bigg(\sum_{\substack{j=1\\i\neq k-1,k}}^{J-1}|v_{i+1,j}^{n+1}-v_{i,j}^{n+1}|^2+\sum_{\substack{i=1\\j\neq l-1,l}}^{J-1}|v_{i,j+1}^{n+1}-v_{i,j}^{n+1}|^2+\frac{1}{\lambda}\sum_{i,j=1}^{J-1}|v_{i,j}^{n+1}|^2\\&+\sum_{j=1}^{J-1}(|v_{k,j}^n-v_{k-1,j}^{n+1}|^2+|v_{k,j}^n-v_{k+1,j}^{n+1}|^2)+\sum_{i=1}^{J-1}(|v_{i,l}^n-v_{i,l-1}^{n+1}|^2+|v_{i,l}^{n+1}-v_{i,l+1}^{n+1}|^2)\\&-\frac{1}{\lambda}\sum_{i,j=1}^{J-1}|G_{i,j}^{n+1}|^2+\frac{1}{\lambda}\sum_{i,j=1}^{J-1}|v_{i,j}^n-G_{i,j}^{n+1}|^2\Bigg)-\lambda h^2\sum_{j=1}^{J-1}(|v_{k,j}^{n+1}|^2-|v_{k,j}^n|^2)\\&-\lambda h^2\sum_{i=1}^{J-1}(|v_{i,l}^{n+1}|^2-|v_{i,l}^n|^2)\leqslant-\lambda h^2\sum_{j=1}^{J-1}(|v_{k,j}^{n+1}|^2-|v_{k,j}^n|^2)-\lambda h^2\sum_{i=1}^{J-1}(|v_{i,l}^{n+1}|^2-|v_{i,l}^n|^2)\\&+\frac{h^2}{2}\sum_{i,j=1}^{J-1}|G_{i,j}^{n+1}|^2,\end{aligned} \tag{5.18}$$

and it follows:

$$\begin{aligned}&\|e_h^{n+1}\|_2^2+2\lambda\tau h^2\sum_{j=1}^{J-1}|v_{k,j}^{n+1}|^2+2\lambda\tau h^2\sum_{i=1}^{J-1}|v_{i,l}^{n+1}|^2\\&\quad\leqslant\|e_h^n\|_2^2+2\lambda\tau h^2\sum_{j=1}^{J-1}|v_{k,j}^n|^2+2\lambda\tau h^2\sum_{i=1}^{J-1}|v_{i,l}^n|^2+\tau h^2\sum_{i,j=1}^{J-1}|G_{i,j}^{n+1}|^2.\end{aligned} \tag{5.19}$$

Hence

$$\|e_h^{n+1}\|_2^2+2\lambda\tau h^2\sum_{j=1}^{J-1}|v_{k,j}^{n+1}|^2+2\lambda\tau h^2\sum_{i=1}^{J-1}|v_{i,l}^{n+1}|^2\leqslant\|e_h^1\|_2^2+2\lambda\tau h^2\sum_{j=1}^{J-1}|v_{k,j}^1|^2+2\lambda\tau h^2\sum_{i=1}^{J-1}|v_{i,l}^1|^2+TC_3^2(\tau+h^2)^2. \tag{5.20}$$

By the inequality similar to (3.23) and (3.24), we have

$$\|e_h^{n+1}\|_2^2\leqslant C(\tau+h^2)^2. \tag{5.21}$$

So the scheme (S3) is unconditional convergent and the convergent order is second order. The similar result holds for the scheme (S4). The proof of Theorem 5.2 is completed.

It is straightforward to extend the results on four subdomains to allow for many subdomains.

6. Numerical results

In this section, we present numerical results examining the stability, accuracy and parallelism of the scheme described above.

6.1. One-dimensional test

For one-dimensional problems, consider the problem (2.1)–(2.3) with the initial value function $U_0(x) = \sin(\pi x)$, and the exact solution $U = \mathrm{e}^{-\pi^2 t}\sin(\pi x)$.

First, we examine the errors in the solution for the scheme (S1). The errors in the solution are presented in Table 6.1. Here the time step $\tau = 1.0\mathrm{e}{-6}$, the rate is the experimental rate of convergence, four mesh refinements are used and two processors are used. It can be seen from this table that the errors appear to be $O(h^2)$.

Next, we examine the stability and parallelism of the scheme (S1). In order to demonstrate the unconditional stability of the scheme, we present the numerical results for $\lambda = 10$, 100, 1000 in Tables 6.2–6.4, respectively. Where $\lambda = \tau/h^2$, CPUs is the number of the processor, T_{all} is the amount of clock time computing 100,000 time steps, S_p is the relative speedup and E_{ff} is the parallel efficiency. When CPUs = 1, this represents the fully implicit solution.

In these runs a uniform mesh is used with 100,000 grid blocks, i.e., $J - 1 = 100{,}000$. The direct solver is used to solve the linear systems on each sub-domain. Table 6.2 shows that, for this test problem, the parallel difference scheme produces results which are the same as the fully implicit scheme, and the amount of clock time needed to solve the problem decreased essentially linearly with increasing the number of processors used. In fact, the speed-up is super-linear. The same phenomena can be seen for $\lambda = 100$ and $\lambda = 1000$ in Tables 6.3

Table 6.1
The accuracy for one-dimensional problem ($\tau = 1.0\mathrm{e}{-6}$, $T = 0.01$)

$J - 1$	10	20	40	80
$\max_{i,n}\lvert u_i^n - U_i^n\rvert$	7.34E−4	1.84E−4	4.64E−5	1.19E−5
$\max_{i,n}\frac{\lvert u_i^n - U_i^n\rvert}{\lvert U_i^n\rvert}$	8.10E−4	2.03E−4	5.12E−5	1.32E−5
Rate	–	2.00	1.99	1.96

Table 6.2
The stability and parallelism for one-dimensional problem ($\lambda = 10$)

CPUs	1	10	20	40	50
$\max_{i,n}\lvert u_i^n - U_i^n\rvert$	4.83E−007	4.83E−007	4.83E−007	4.83E−007	4.83E−007
T_{all} (s)	4564	337	173	91	74
S_p	1	13.52	26.27	49.78	61.32
E_{ff} (100%)	1	1.35	1.31	1.24	1.23

Table 6.3
The stability and parallelism for one-dimensional problem ($\lambda = 100$)

CPUs	1	10	20	40	50
$\max_{i,n}\lvert u_i^n - U_i^n\rvert$	4.88E−007	4.88E−007	4.88E−007	4.88E−007	4.88E−007
T_{all} (s)	4553	339	173	92	75
S_p	1	13.40	26.20	49.28	60.22
E_{ff} (100%)	1	1.34	1.31	1.23	1.20

Table 6.4
The stability and parallelism for one-dimensional problem ($\lambda = 1000$)

CPUs	1	10	20	40	50
$\max_{i,n}\lvert u_i^n - U_i^n\rvert$	4.50E−007	4.50E−007	4.50E−007	4.50E−007	4.50E−007
T_{all} (s)	4481	340	174	93	74
S_p	1	13.16	25.73	47.88	60.15
E_{ff} (100%)	1	1.32	1.29	1.20	1.20

and 6.4, respectively. Hence, the parallel scheme (S1) is stable for $\lambda = 10$, 100 and 1000. Notice that the explicit scheme of this test problem is stable for $\lambda \leqslant \frac{1}{2}$. These results indicate that the scheme is unconditional stable.

6.2. Two-dimensional test

Consider the following two-dimensional problem:

$$u_t = u_{xx} + u_{yy} + f(x,y,t), \quad (x,y) \in \Omega, \ t \in (0,T].$$

The solution is chosen to be

$$u(x,y,t) = \mathrm{e}^{-2\pi^2 t}(2 + \sin(\pi x)\sin(\pi y)).$$

The initial function is $u(x,y,0) = 2 + \sin(\pi x)\sin(\pi y)$, $f(x,y,t) = -4\pi^2\mathrm{e}^{-2\pi^2 t}$, and the Dirichlet boundary conditions are used.

First, we examine the errors in the solution for the parallel scheme (S3). The errors in the solution are presented in Table 6.5. Here the rate is the experimental rate of convergence, four mesh refinements are used and four processors (2×2) are used. We use the diagonally preconditioned conjugate gradient algorithm in [12] to solve the linear systems on each sub-domains. As can be seen in this table, the errors appear to be $O(h^2)$.

Next, we examine the stability of the scheme for the two-dimensional problem. In order to demonstrate the unconditional stability of the scheme, we present the numerical results for different λ in Table 6.6. Here $\tau = 1.0\mathrm{e}{-4}$, $T = 0.1$, and four processors are used. This table shows that, for this test problem, the parallel difference scheme produces good results for different λ. Notice that the stability constraint for fully explicit scheme is $\lambda \leqslant 1/4$, our parallel difference scheme is still stable for $\lambda = 400$. These indicate that the scheme is unconditional stable.

At last, we examine the parallelism of the scheme for the two-dimensional problem. In Fig. 6.1, we present the speed-up for different number of processor. In these runs a uniform mesh is used with 1000 grid blocks in each direction, i.e., the mesh scale is 1000×1000. The time step $\tau = 1.0\mathrm{e}{-5}$, $T = 0.01$ and $\lambda = 10$. A diagonally preconditioned conjugate gradient algorithm is used to solve the linear systems on each sub-domains. The x-direction and y-direction have the same number of processor m, i.e., the number of processor is $m \times m$. Notice that the global problem is fully decoupled by using our parallel difference scheme, hence, some small scale systems on each sub-domain can be formed. No communication between different sub-domains is necessary to solve the linear systems on sub-domain. And the communication only exists between the neighboring processors in order to update the interface values, i.e., the communication is local. So our scheme has high parallelism.

To show the performance of our new parallel difference schemes for two-dimensional problems, we will compare them with the well-known parallelization method, the so-called parallel algebraic method, which uses the parallel preconditioned conjugate gradient method to solve the global linear algebraic system arising from the fully implicit scheme on the global space domain. That is, the standard fully implicit discretization for the parabolic equations is used, and then the parallel algebraic solver (parallel preconditioned conjugate gradient

Table 6.5
The accuracy for two-dimensional problem ($\tau = 1.0\mathrm{e}{-6}$, $T = 0.01$)

$(J-1)\times(J-1)$	10×10	20×20	40×40	80×80
$\max_{i,j,n}\lvert U^n_{i,j} - u^n_{i,j}\rvert$	1.33E−3	3.38E−4	8.84E−5	2.59E−5
$\max_{i,j,n}\frac{\lvert U^n_{i,j}-u^n_{i,j}\rvert}{\lvert U^n_{i,j}\rvert}$	5.42E−4	1.37E−4	3.59E−5	1.05E−5
Rate	–	1.97	1.93	1.77

Table 6.6
The stability for two-dimensional problem ($\tau = 1.0\mathrm{E}{-4}$, $T = 0.1$)

λ	1	16	25	64	100	400
$\max_{i,j,n}\lvert U^n_{i,j} - u^n_{i,j}\rvert$	8.88E−4	8.49E−4	1.39E−3	3.00E−3	4.08E−3	9.48E−3
$\max_{i,j,n}\frac{\lvert U^n_{i,j}-u^n_{i,j}\rvert}{\lvert U^n_{i,j}\rvert}$	1.66E−3	1.50E−3	2.50E−3	5.50E−3	7.50E−3	1.75E−2

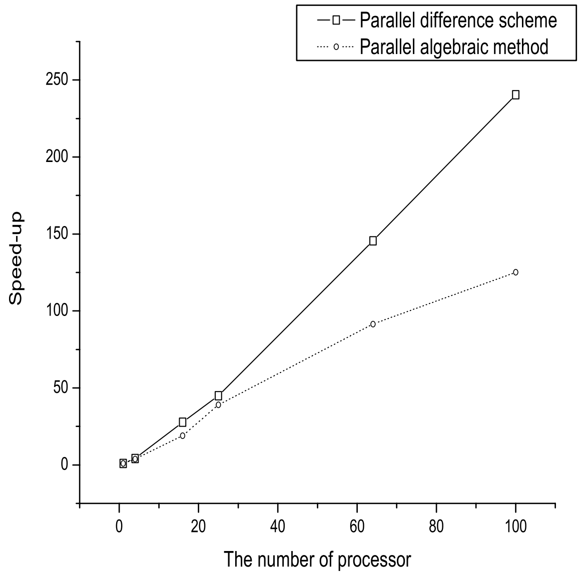

Fig. 6.1. The parallelism for two-dimensional problem.

Table 6.7
The average number of iteration for parallel difference scheme

CPUs	1	4	16	25	64	100
$it^{\#}$	83	82	81	80	78	74

method, see [12]) is exploited to solve the linear system in parallel. The parallel algebraic method needs the local communications between the neighboring processors and the global communications among all processors as well. Hence, its parallelism is low compared with our method, especially in the cases of CPUs number being increased. The same criterion is used to decide whether the solution is convergence for the parallel difference method and parallel algebraic method.

In Fig. 6.1, the solid line expresses our method and the dot line expresses the parallel algebraic method. As can be seen in this figure, the speed-up of our method is better than parallel algebraic method. Moreover, the speed-up is over 240 when 100 processors were used. This is because that the iterative number of small scale system is less than the large scale system. In Table 6.7, we present the iterative number of parallel difference scheme, where $it^{\#}$ is the average number of iteration for solving linear system. For parallel algebraic method, the iterative number is same as the case of parallel difference scheme when one processor is used, i.e., the iterative number is 83, and there is no difference for different number of processor. When 100 processors are used, the iterative number of parallel algebraic method is 83, however the iterative number of parallel difference scheme is 74, hence our method converges faster.

7. Generalizations and conclusions

In this paper, the fixed time step be used. In fact variable time step can be used. Next we take the scheme (S1) as an example, and extend it to variable time step. The scheme with variable time step is as follows:

$$\Delta_{\tau_{n+1}} u_j^{n+1} = \delta^2 u_j^{n+1}, \quad j = 1, 2, \ldots, k-2, k+2, \ldots, J-1,$$

$$\Delta_{\tau_{n+1}} u_{k-1}^{n+1} = \frac{1}{h^2}\left(\left(u_k^n - \frac{\tau_{n+1}}{\tau_n}(u_k^n - u_k^{n-1})\right) - u_{k-1}^{n+1} - (u_{k-1}^{n+1} - u_{k-2}^{n+1})\right),$$

$$\Delta_{\tau_{n+1}} u_{k+1}^{n+1} = \frac{1}{h^2}\left(u_{k+2}^{n+1} - u_{k+1}^{n+1} - \left(u_{k+1}^{n+1} - \left(u_k^n - \frac{\tau_{n+1}}{\tau_n}(u_k^n - u_k^{n-1})\right)\right)\right).$$

It is equivalent to

$$\Delta_{\tau_{n+1}} u_j^{n+1} = \delta^2 u_j^{n+1}, \quad j = 1, 2, \ldots, k-2, k, k+2, \ldots, J-1,$$
$$\Delta_{\tau_{n+1}} u_{k-1}^{n+1} = \delta^2 u_{k-1}^{n+1} - \lambda_{n+1} \Delta_{\tau_{n+1}} u_k^{n+1} + \lambda_{n+1} \Delta_{\tau_n} u_k^n,$$
$$\Delta_{\tau_{n+1}} u_{k+1}^{n+1} = \delta^2 u_{k+1}^{n+1} - \lambda_{n+1} \Delta_{\tau_{n+1}} u_k^{n+1} + \lambda_{n+1} \Delta_{\tau_n} u_k^n,$$

where $\Delta_{\tau_{n+1}} u_j^{n+1} = \frac{u_j^{n+1} - u_j^n}{\tau_{n+1}}$, $\tau_{n+1} = t^{n+1} - t^n$, $\lambda_{n+1} = \frac{\tau_{n+1}}{h^2}$.

The other schemes (S2)–(S4) can also be extended to the schemes with variable time step. We can obtain the similar theorem of stability and convergence for those schemes with variable time step.

The procedure constructing parallel difference scheme is clearly generalizable to three-dimensional space.

To conclude, in this paper we have presented a parallel difference scheme with unconditional stability and second order accuracy for linear parabolic equation. The design of the schemes is simple, and it can be implemented with little extra effort by modifying the original sequential codes. The numerical results demonstrate the good performance of the parallel schemes, i.e., they are unconditionally stable, and have second order accuracy and high degree of parallelism. In particular, the super-linear speedup is achieved.

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