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Parallel difference schemes with interface extrapolation terms for quasi-linear parabolic systems

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Abstract In this paper some new parallel difference schemes with interface extrapolation terms for a quasi-linear parabolic system of equations are constructed. Two types of time extrapolations are proposed to give the interface values on the interface of sub-domains or the values adjacent to the interface points, so that the unconditional stable parallel schemes with the second accuracy are formed. Without assuming heuristically that the original boundary value problem has the unique smooth vector solution, the existence and uniqueness of the discrete vector solutions of the parallel difference schemes constructed are proved. Moreover the unconditional stability of the parallel difference schemes is justified in the sense of the continuous dependence of the discrete vector solution of the schemes on the discrete known data of the original problems in the discrete $W_2^{(2,1)}(Q_\triangle)$ norms. Finally the convergence of the discrete vector solutions of the parallel difference schemes with interface extrapolation terms to the unique generalized solution of the original quasi-linear parabolic problem is proved. Numerical results are presented to show the good performance of the parallel schemes, including the unconditional stability, the second accuracy and the high parallelism.

Keywords: parallel difference scheme, interface extrapolation, quasi-linear parabolic system, unconditional stability, convergence.

MSC(2000): 65M06, 65M12, 65M55

1 Introduction

Many kinds of unconditional stable parallel difference schemes for the parabolic problem are constructed in [1–5]. The alternating group explicit (AGE) method for the diffusion equation using the Saul'yev' scheme is constructed in [1]. The alternating segment explicit–implicit (ASE–I) method in [2] is designed. The pure alternating segment explicit–implicit (PASE–I) method is constructed in [3]. These schemes in [1–3] use the alternating explicit–implicit technique in the time direction, which not only bring some difficulties in implementing these parallel schemes in applications and bring a new computational complexity, but also up to now make the rigorous theory of the alternating explicit–implicit schemes for general nonlinear parabolic equations not established completely, e.g. there is lack of rigorous analysis of stability and convergence. In [4] a general modified alternating difference scheme for the linear and nonlin-

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ear problem is constructed, and the unconditional stability of these schemes is proved. The alternating group explicit scheme, alternating segment explicit–implicit scheme and alternating segment Crank–Nicolson scheme are the special cases of the general modified alternating difference scheme, and many new parallel schemes with the unconditional stability can be obtained by taking some specific parameters in the general parallel schemes. In particular, the explicit scheme in the sub-domains is not needed on each time layer, but the alternating movement of the inner interface between sub-domains is needed on the odd–even time layers. These schemes above are essentially three–layer schemes, and have certain difficulties to be implemented in practice for the multi–dimensional problem. The parallel scheme in [5] uses the implicit scheme in the sub–domains and the Du Fort–Frankel scheme at the inner interface, which is also a three–layer explicit scheme.

There are also many researches on the two–layer schemes with unconditional or conditional stability. A parallel scheme has been proposed in [6], where instead of using the same spacing h as for the interior points where the implicit scheme is applied, a larger spacing H_D is used at each interface point where the explicit scheme is applied. There are also some other schemes with the domain decomposition in [7, 8]. These schemes are conditionally stable. In [9] some specific parallel difference schemes are constructed for the parabolic problem. After dividing the domain, these parallel schemes take the values of the previous time layer as the boundary condition of sub-domains problems, and then parallel computation can be implemented. These schemes are proved to be unconditionally stable and convergent in the sense of L^∞ norm and H^1 norm. Although the truncation error of these schemes is $O(1)$ at the interface point between the sub-domains, the convergent order of the difference solution is $O(\tau + h)$. The method of taking the values of the previous time layer near the inner interface point has been generalized to the quasi–linear parabolic problems and two dimensional semilinear parabolic problems. These parallel schemes are proved to be unconditionally stable and convergent, but the convergent order is only one order. In order to avoid the loss of accuracy near the interface point, the parallel iterative difference schemes based on the interface correction are constructed in [10], in which the main idea is that the values at the interface points are corrected unceasingly or periodically by the implicit scheme at the (nonlinear) iterative process of computing the values in the sub-domains in order to get the unconditional stability and the second order accuracy. No one of these parallel difference schemes can at one time satisfy the three requirements: (i) the unconditional stability, (ii) the second order convergence, (iii) easy implementation.

In this paper we will construct some parallel schemes which satisfy all of the following six requirements presented in [10]: (i) the unconditional stability, (ii) the second order convergence, (iii) the high degree of parallelism, (iv) it can be used readily to modify the serial program into the parallel program by putting the parallel scheme in practice, (v) the schemes constructed are simple and can be easily implemented in the parallel computer, (vi) the conditional number of algebraic systems resulting from the parallel schemes is relatively low in the sub-domain and the highly efficient solver of the algebraic systems can be used.

In order to achieve the goal of getting a parallel scheme with both the unconditional stability and the second order accuracy, we introduce a new method based on the time extrapolation in this paper. We take the linear combination of values of previous two time layers at the inner interface among the sub-domains as the (Dirichlet) boundary condition of the sub-domain problems in order to solve simultaneously the sub-domain problems. This method can guarantee that the truncation error of the difference scheme near the interface point is still $O(\tau + h^2)$, which accords with the truncation error of the implicit scheme in the sub-domains. Hence the accuracy of the difference solution near the interface point is improved compared with the previous parallel schemes. Thus the convergent order of the difference solution in the whole domain is $O(\tau + h^2)$.

The general parallel difference schemes with interface extrapolation terms include, among others, two new parallel schemes. In one of them, to obtain the interface values on the interface of sub-domains an explicit scheme of the Jacobian type is employed, and then the fully implicit scheme is used in the sub-domains. Here, in the explicit scheme of the Jacobian type, the values at the points being adjacent to the interface points are taken as the linear combination of values of previous two time layers at the adjoining points of the inner interface. For the construction of another new parallel difference scheme, the main procedure is as follows. Firstly the linear combination of values of previous two time layers at the interface points among the sub-domains is used as the (Dirichlet) boundary condition for solving the sub-domain problems. Then the values in the sub-domains are calculated by the fully implicit scheme. And finally the interface values are computed by the fully implicit scheme, and in fact these calculations of the last step are explicit since the values adjacent to the interface points have been obtained in the previous step.

In sec. 2, a kind of new parallel difference schemes with the interface extrapolation for the quasi–linear parabolic systems is constructed. In sec. 3, the priori estimate of the discrete solution of these schemes is proved without the assumption that the unique smooth vector solution of the original problems for the quasi–linear parabolic systems exists, hence we can get the existence of the difference solution from the fixed point theorem. In secs. 4 and 5, the uniqueness and the unconditional stability of the difference solution are proved. In sec. 6, we prove that the difference solution converges to the unique generalized solution of the original quasi–linear parabolic equations. In sec. 7, numerical experiments are presented to verify the theoretical results. Moreover it is shown that the super–linear speedup is achieved. In the last section, some remarks and future plans are given.

2 Difference scheme with interface extrapolation

2.1 Let us now consider the quasi–linear parabolic problem with the initial boundary value,

$$u_t = A(x,t,u)u_{xx} + f(x,t,u,u_x), \ \ 0 < x < l, 0 < t \leqslant T \tag{2.1}$$

$$u(0,t) = u(l,t) = 0, 0 < t \leqslant T \tag{2.2}$$

$$u(x,0) = \varphi(x), 0 \leqslant x \leqslant l \tag{2.3}$$

where $u(x,t) = (u_1(x,t), \dots, u_m(x,t))$ is the m-dimensional vector unknown function ($m \geqslant 1$), $u_t = \frac{\partial u}{\partial t}, u_x = \frac{\partial u}{\partial x}$ and $u_{xx} = \frac{\partial^2 u}{\partial x^2}$ are the corresponding vector derivatives. $A(x,t,u)$ is an $m \times m$ positive definite coefficient matrix, $f(x,t,u,p)$ and $\varphi(x)$ are the m-dimensional vector functions. Denote $Q_T = \{0 \leqslant x \leqslant l, 0 \leqslant t \leqslant T\}$, where $l > 0, T > 0$.

Suppose that the following conditions are fulfilled.

(I) $A(x,t,u)$ is continuous with respect to $(x,t) \in Q_T$ and $u \in R^m$, and is uniformly

Lipschitz continuous for $u \in R^m$. There exists a positive constant $\sigma_0 > 0$ such that for any $\xi \in R^m$, $(x,t) \in Q_T$ and $u \in R^m$,

$$(\xi, A(x,t,u)\xi) \geqslant \sigma_0 |\xi|^2.$$

(II) $f(x,t,u,p)$ is continuous with respect to $(x,t) \in Q_T$ and $u, p \in R^m$, and is uniformly Lipschitz continuous for $u, p \in R^m$. Then there exists a positive constant $C > 0$ such that $|f(x,t,u,p)| \leqslant |\bar{f}(x,t)| + C(|u| + |p|)$ for any $(x,t) \in Q_T$ and $u, p \in R^m$, where $\bar{f}(x,t) = f(x,t,0,0)$.

(III) $\varphi(x) \in H^1[0,l]$ is an m-dimensional vector function satisfying $\varphi(0) = \varphi(l) = 0$.

2.2 Divide the domain Q_T into small grids by $x = x_j$ $(j = 0,1,\ldots,J)$ and $t = t^n$ $(n = 0,1,\ldots,N)$, where $x_j = jh$, $t^n = n\tau$, J and N are the positive integers, h and τ are the step lengths of the grids. Denote $Q_j^n = \{x_j < x \leqslant x_{j+1}, t^n < t \leqslant t^{n+1}\}$, where $j = 0,1,\ldots,J-1; n = 0,1,\ldots,N-1$. Denote $v_\Delta = v_h^\tau = \{v_j^n | j = 0,1,\ldots,J; n = 0,1,\ldots,N\}$ as the m–dimensional discrete vector function defined on the discrete rectangular domain $Q_\Delta = \{(x_j, t^n) | j = 0,1,\ldots,J; n = 0,1,\ldots,N\}$ of the grid points.

Denote $r = \frac{\tau}{h^2}$, $\Delta_\tau v_j^{n+1} = \frac{v_j^{n+1} - v_j^n}{\tau}$, $\delta v_{j+\frac{1}{2}}^n = \frac{v_{j+1}^n - v_j^n}{h}$, and for $n \geqslant 1$,

$$\overset{*}{\delta}{}^2 v_j^{n+1} = \frac{1}{h^2}(v_{j+1}^{\bar{n}+\lambda_j} - 2v_j^{n+1} + v_{j-1}^{\bar{n}+\mu_j}),$$

where $v_{j+1}^{\bar{n}+\lambda_j} = \lambda_j v_{j+1}^{n+1} + (1-\lambda_j)(2v_{j+1}^n - v_{j+1}^{n-1}), v_{j-1}^{\bar{n}+\mu_j} = \mu_j v_{j-1}^{n+1} + (1-\mu_j)(2v_{j-1}^n - v_{j-1}^{n-1})$. If $\lambda_j = \mu_j = 0$ for some j, then a time extrapolation is used in the construction of $\overset{*}{\delta}{}^2 v_j^{n+1}$. When $n = 0$, define

$$\overset{*}{\delta}{}^2 v_j^{n+1} = \delta^2 v_j^1 = \frac{1}{h^2}(v_{j+1}^1 - 2v_j^1 + v_{j-1}^1).$$

Construct the general parallel difference schemes with the interface extrapolation for the quasi–linear parabolic systems (2.1)–(2.3) as follows:

$$\frac{v_j^{n+1} - v_j^n}{\tau} = A_j^{n+1} \overset{*}{\delta}{}^2 v_j^{n+1} + f_j^{n+1}, \quad (j = 1,2,\ldots,J-1; n = 0,1,\ldots,N-1); \tag{2.4}$$

$$v_0^{n+1} = v_J^{n+1} = 0, \quad (n = 0,1,\ldots,N-1), \tag{2.5}$$

$$v_j^0 = \varphi_j, \quad (j = 0,1,\ldots,J), \tag{2.6}$$

where

$$\begin{aligned} &\varphi_j = \varphi(x_j), (j = 0,1,\ldots,J), \quad \varphi_0 = \varphi_J = 0; \\ &A_j^{n+1} = A(x_j, t^{n+1}, \bar{\delta}^0 v_j^{n+1}), \quad f_j^{n+1} = f(x_j, t^{n+1}, \hat{\delta}^0 v_j^{n+1}, \bar{\delta}^1 v_j^{n+1}). \end{aligned}$$

For $n \geqslant 1$, define $\bar{\delta}^0 v_j^{n+1}$, $\hat{\delta}^0 v_j^{n+1}$, $\bar{\delta}^1 v_j^{n+1}$ as follows:

$$\begin{aligned} \bar{\delta}^0 v_j^{n+1} = {} & \lambda_j \alpha_{1j}^n v_{j+1}^{n+1} + \alpha_{2j}^n v_j^{n+1} + \mu_j \alpha_{3j}^n v_{j-1}^{n+1} + \alpha_{4j}^n v_{j+1}^n + \alpha_{5j}^n v_j^n + \alpha_{6j}^n v_{j-1}^n \\ & + \alpha_{7j}^n v_{j+1}^{n-1} + \alpha_{8j}^n v_j^{n-1} + \alpha_{9j}^n v_{j-1}^{n-1}, \\ \hat{\delta}^0 v_j^{n+1} = {} & \lambda_j \beta_{1j}^n v_{j+1}^{n+1} + \beta_{2j}^n v_j^{n+1} + \mu_j \beta_{3j}^n v_{j-1}^{n+1} + \beta_{4j}^n v_{j+1}^n + \beta_{5j}^n v_j^n + \beta_{6j}^n v_{j-1}^n \\ & + \beta_{7j}^n v_{j+1}^{n-1} + \beta_{8j}^n v_j^{n-1} + \beta_{9j}^n v_{j-1}^{n-1}, \end{aligned}$$

$$\begin{aligned}\bar{\delta}^1 v_j^{n+1} &= \gamma_{1j}^n \frac{v_{j+1}^{\bar{n}+\lambda_j} - v_j^{n+1}}{h} + \gamma_{2j}^n \frac{v_j^{n+1} - v_{j-1}^{\bar{n}+\mu_j}}{h} \\ &\quad + \gamma_{3j}^n \delta v_{j+\frac{1}{2}}^n + \gamma_{4j}^n \delta v_{j-\frac{1}{2}}^n + \gamma_{5j}^n \delta v_{j+\frac{1}{2}}^{n-1} + \gamma_{6j}^n \delta v_{j-\frac{1}{2}}^{n-1}.\end{aligned} \tag{2.7}$$

For $n = 0$, define $A_j^1 = A(x_j, t^1, \bar{\delta}^0 v_j^1)$, $f_j^1 = f(x_j, t^1, \hat{\delta}^0 v_j^1, \bar{\delta}^1 v_j^1)$, where

$$\begin{aligned}\bar{\delta}^0 v_j^1 &= \alpha_{1j}^0 v_{j+1}^1 + \alpha_{2j}^0 v_j^1 + \alpha_{3j}^0 v_{j-1}^1 + \alpha_{4j}^0 v_{j+1}^0 + \alpha_{5j}^0 v_j^0 + \alpha_{6j}^0 v_{j-1}^0, \\ \hat{\delta}^0 v_j^1 &= \beta_{1j}^0 v_{j+1}^1 + \beta_{2j}^0 v_j^1 + \beta_{3j}^0 v_{j-1}^1 + \beta_{4j}^0 v_{j+1}^0 + \beta_{5j}^n v_j^0 + \beta_{6j}^n v_{j-1}^0, \\ \bar{\delta}^1 v_j^1 &= \gamma_{1j}^0 \delta v_{j+\frac{1}{2}}^1 + \gamma_{2j}^0 \delta v_{j-\frac{1}{2}}^1 + \gamma_{3j}^0 \delta v_{j+\frac{1}{2}}^0 + \gamma_{4j}^0 \delta v_{j-\frac{1}{2}}^n.\end{aligned}$$

2.3 The following assumption will be needed.

(IV) Suppose that $r = \frac{\tau}{h^2} \leqslant \Lambda$ holds for τ and h being small enough, where Λ is a given positive constant. For $1 \leqslant j \leqslant J-1$, there are $0 \leqslant \lambda_j, \mu_j \leqslant 1$, $\lambda_j + \mu_{j+1} \geqslant 1$, and for a constant $C > 0$ the coefficients in (2.7) satisfy

$$\begin{aligned}&\lambda_j \alpha_{1j}^n + \alpha_{2j}^n + \mu_j \alpha_{3j}^n + \alpha_{4j}^n + \alpha_{5j}^n + \alpha_{6j}^n + \alpha_{7j}^n + \alpha_{8j}^n + \alpha_{9j}^n = 1, \\ &\lambda_j \beta_{1j}^n + \beta_{2j}^n + \mu_j \beta_{3j}^n + \beta_{4j}^n + \beta_{5j}^n + \beta_{6j}^n + \beta_{7j}^n + \beta_{8j}^n + \beta_{9j}^n = 1, \\ &\gamma_{1j}^n + \gamma_{2j}^n + \gamma_{3j}^n + \gamma_{4j}^n + \gamma_{5j}^n + \gamma_{6j}^n = 1, \\ &|\lambda_j \alpha_{1j}^n| + |\alpha_{2j}^n| + |\mu_j \alpha_{3j}^n| + |\alpha_{4j}^n| + |\alpha_{5j}^n| + |\alpha_{6j}^n| + |\alpha_{7j}^n| + |\alpha_{8j}^n| + |\alpha_{9j}^n| \leqslant C, \\ &|\lambda_j \beta_{1j}^n| + |\beta_{2j}^n| + |\mu_j \beta_{3j}^n| + |\beta_{4j}^n| + |\beta_{5j}^n| + |\beta_{6j}^n| + |\beta_{7j}^n| + |\beta_{8j}^n| + |\beta_{9j}^n| \leqslant C, \\ &|\gamma_{1j}^n| + |\gamma_{2j}^n| + |\gamma_{3j}^n| + |\gamma_{4j}^n| + |\gamma_{5j}^n| + |\gamma_{6j}^n| \leqslant C.\end{aligned}$$

For $n = 0$, suppose that there hold similar conditions.

Remark. The constants α_j^n, β_j^n and γ_j^n depend on $j = 1, 2, \ldots, J-1$ and $n = 0, 1, \ldots, N-1$, λ_j and μ_j depend on $j = 1, 2, \ldots, J-1$. They may be different for different j and n. The scheme includes many difference schemes with the intrinsic parallelism, for example, the resulting scheme when $\lambda_j = 0$ or $\mu_j = 0$ at some grids.

2.4 Let $1 < k < J-1$. When $\lambda_k = 0$, it is obvious that $\{v_j^{n+1} | j \leqslant k\}$ in the left sub-domain $\{x_j | j \leqslant k\}$ is independent of $\{v_j^{n+1} | j > k\}$ in the right sub-domain $\{x_j | j > k\}$. Similarly, when $\mu_k = 0$, $\{v_j^{n+1} | j \geqslant k\}$ in the right sub-domain $\{x_j | j \geqslant k\}$ is independent of $\{v_j^{n+1} | j < k\}$ in the left sub-domain $\{x_j | j < k\}$. Hence we can implement the parallel computation as long as we let $\lambda_j = 0$ or $\mu_j = 0$. For example, taking $\lambda_k = \mu_{k+2} = 0$ for some k $(2 < k < J-2)$ and $\lambda_j = \mu_j = 1$ for others, we get a new parallel scheme. That is, we take the extrapolation value $2v_{k+1}^n - v_{k+1}^{n-1}$ as the inner boundary condition at the point $j = k+1$, and use the implicit scheme in the sub-domains $\{j < k+1\}$ and $\{j > k+1\}$ respectively, and then use the implicit scheme at the point $j = k+1$ to get v_{k+1}^{n+1} (by using of the values of v_k^{n+1} and v_{k+2}^{n+1} obtained).

In order to further understand the construction of (2.4)–(2.6), we specially consider the parallel scheme for $u_t = u_{xx}$. First the values of $\{v_j^{n+1} | j \neq k\}$ in the sub-domains $\{x_j | j < k\}$ and $\{x_j | j > k\}$ are computed by using the following schemes:

$$\begin{aligned}\Delta_\tau v_{k-1}^{n+1} &= \frac{1}{h^2}((2v_k^n - v_k^{n-1}) - 2v_{k-1}^{n+1} + v_{k-2}^{n+1}), \\ \Delta_\tau v_j^{n+1} &= \frac{1}{h^2}(v_{j+1}^{n+1} - 2v_j^{n+1} + v_j^{n+1}), \quad j \neq k-1, k, k+1,\end{aligned}$$

$$\Delta_\tau v_{k+1}^{n+1} = \frac{1}{h^2}(v_{k+2}^{n+1} - 2v_{k+1}^{n+1} + (2v_k^n - v_k^{n-1})),$$

Then v_k^{n+1} is computed using the following scheme:

$$\Delta_\tau v_k^{n+1} = \frac{1}{h^2}(v_{k+1}^{n+1} - 2v_k^{n+1} + v_k^{n+1}).$$

This scheme is a specific example, where $\lambda_{k-1} = 0, \mu_{k+1} = 0$, and $\lambda_j = 1$ for any $j \neq k-1$, and $\mu_j = 1$ for any $j \neq k+1$ in (2.4). Notice that $\Delta_\tau v_k^n = \delta^2 v_k^n$. This scheme can be rewritten as

$$\begin{aligned}
\Delta_\tau v_{k-1}^{n+1} &= \frac{1}{h^2}(v_k^n - 2v_{k-1}^{n+1} + v_{k-2}^{n+1}) + r\delta^2 v_k^n,\\
\Delta_\tau v_j^{n+1} &= \frac{1}{h^2}(v_{j+1}^{n+1} - 2v_j^{n+1} + v_j^{n+1}), \quad j \neq k-1, k+1,\\
\Delta_\tau v_{k+1}^{n+1} &= \frac{1}{h^2}(v_{k+2}^{n+1} - 2v_{k+1}^{n+1} + v_k^n) + r\delta^2 v_k^n,
\end{aligned}$$

It is obvious that this scheme is a two-layer scheme, and can implement the parallel computation in nature. Although the scheme (2.4) involves the three-layer values at an inner interface, in fact it can be reduced to be a certain two-layer scheme. Some examples of constructing specific parallel schemes are presented in sec. 7.

2.5 For the discrete function $u_h = \{u_j | j = 0, 1, \ldots, J\}$, where $u_0 = u_J = 0$, define the discrete norm as follows:

$$\|u_h\|_\infty = \max_{0 \leqslant j \leqslant J} |u_j|, \quad \|u_h\|_2^2 = \sum_{j=1}^{J-1} |u_j|^2 h, \quad \|\delta u_h\|_2^2 = \sum_{j=0}^{J-1} |\delta u_{j+\frac{1}{2}}|^2 h.$$

We need some lemmas (see [11]) as follows:

Lemma 2.1 (The discrete Green formula). *Let u_j and v_j be the discrete function defined on $\{x_j | j = 0, 1, \ldots, J\}$, then*

$$\sum_{j=0}^{J-1} u_j(v_{j+1} - v_j) = -\sum_{j=1}^{J-1} (u_j - u_{j-1}) v_j - u_0 v_0 + u_{J-1} v_J.$$

Lemma 2.2 (The discrete Gronwall inequality). (i) *Let $w^n \geqslant 0$ be a discrete function defined on $\{t^n | n = 0, 1, \ldots, N\}$, and satisfy*

$$w^{n+1} - w^n \leqslant B\tau(w^{n+1} + w^n) + C_n \tau, \quad n = 0, 1, \ldots, N-1,$$

where B and C_n are nonnegative constant, $N\tau = T$. Then

$$w^n \leqslant \left(w^0 + \sum_{k=0}^{N} C_k \tau\right) e^{4BT}, \quad n = 0, 1, \ldots, N,$$

where τ satisfies $4B\tau \leqslant \frac{N-1}{N}$.

(ii) *Suppose that the discrete function $w^\tau = \{w^n \geqslant 0 | n = 0, 1, \ldots, N\}$, $N\tau = T$, satisfies*

$$w^n \leqslant B + C \sum_{k=0}^{n} w^k \tau,$$

where B *and* C *are nonnegative constant. Then*

$$w^n \leqslant B(e^{2CT}+1),$$

where τ *is chosen such that* $C\tau \leqslant \frac{1}{2}$.

(iii) *Let* $w^n \geqslant 0$ *be a discrete function defined on* $\{t^n|n=0,1,\ldots,N\}$, *and satisfy*

$$w^{n+1}-w^n \leqslant B\tau(w^{n-1}+w^n+w^{n+1})+C_n\tau, \ \ n=1,\ldots,N-1$$

where B *and* C_n *are nonnegative constant,* $N\tau=T$. *Then, when* $n\geqslant 1$,

$$w^{n+1}\leqslant 3B\sum_{k=0}^{n+1}w^k\tau+G^n, \ \ n=1,\ldots,N-1,$$

and

$$w^{n+1}\leqslant (3B(w^0+w^1)\tau+2G^n)e^{6BT}, \ \ n=1,\ldots,N-1,$$

where $G^n=w^1+\sum_{k=1}^n C_k\tau$, *and* τ *is assumed to satisfy* $2B\tau\leqslant 1$.

We only give the proof of (iii) here. Denote

$$F^{n+1}=\sum_{k=0}^{n+1}w^k\tau, \ \ G^n=\sum_{k=1}^{n}C_k\tau+w^1.$$

There holds $w^{n+1}\leqslant 3BF^{n+1}+G^n$. Hence

$$\begin{aligned}F^{n+1}&\leqslant \frac{1}{1-3B\tau}F^n+\frac{\tau}{1-3B\tau}G^n\\&\leqslant \frac{1}{(1-3B\tau)^n}F^1+\frac{\tau}{(1-3B\tau)^{n-1}}G^2+\cdots+\frac{\tau}{1-3B\tau}G^n\\&\leqslant \frac{1}{(1-3B\tau)^n}(w^1+w^0)\tau+\frac{1}{3B}(\frac{\tau}{(1-3B\tau)^{n-1}}-1)G^n\\&\leqslant\left((w^0+w^1)\tau+\frac{G^n}{3B}\right)e^{6BT}.\end{aligned}$$

So the proof of (iii) is completed.

Lemma 2.3(The interpolation formula). *For any discrete function* $u_h=\{u_j|j=0,1,\ldots,J\}$ ($Jh=l$), *the following assertions hold.*

(i) *For all* $\varepsilon>0$, *there is*

$$\|u_h\|_\infty^2\leqslant \varepsilon\|\delta u_h\|_2^2+\left(1+\frac{C}{\varepsilon}\right)\|u_h\|_2^2, \tag{2.8}$$

where C *is a constant depending on* l, *and independent of* ε, h *and* u_h.

(ii) *If* $u_0=u_J=0$, *then*

$$\|u_h\|_2\leqslant l\|\delta u_h\|_2, \ \ \|u_h\|_\infty\leqslant\|\delta u_h\|_2^{\frac{1}{2}}\|u_h\|_2^{\frac{1}{2}}. \tag{2.9}$$

(iii) *There exists a constant* C *independent of* h *and* l, *such that*

$$\|\delta u_h\|_2\leqslant C(\|u_h\|_2^{\frac{1}{2}}\|\delta^2 u_h\|_2^{\frac{1}{2}}+l^{-1}\|u_h\|_2). \tag{2.10}$$

In this paper, C is a constant independent of h and τ, and it may be different at different places. Denote

$$\|\overset{*}{\delta}{}^{2}v_h^{n+1}\|_2^2=\sum_{j=1}^{J-1}|\overset{*}{\delta}{}^{2}v_j^{n+1}|^2h,\ \ \|\Delta_\tau v_h^{n+1}\|_2^2=\sum_{j=1}^{J-1}|\Delta_\tau v_j^{n+1}|^2h.$$

3 Priori estimate and existence

3.1 We will prove the existence of the discrete solution of the parallel scheme (2.4)–(2.6) in this section. First the priori estimate of the discrete solution is derived.

Making the scalar product of $\overset{*}{\delta}{}^{2}v_j^{n+1}h\tau$ with (2.4) and summing up the resulting products for $j=1,2,\ldots,J-1$, we get

$$\sum_{j=1}^{J-1}(\overset{*}{\delta}{}^{2}v_j^{n+1},v_j^{n+1}-v_j^n)h=\tau\sum_{j=1}^{J-1}(\overset{*}{\delta}{}^{2}v_j^{n+1},A_j^{n+1}\,\overset{*}{\delta}{}^{2}v_j^{n+1}+f_j^{n+1})h. \tag{3.1}$$

Obviously, there are

$$\overset{*}{\delta}{}^{2}v_j^{n+1}=\delta^2v_j^{n+1}-r[(1-\lambda_j)(\Delta_\tau v_{j+1}^{n+1}-\Delta_\tau v_{j+1}^n)+(1-\mu_j)(\Delta_\tau v_{j-1}^{n+1}-\Delta_\tau v_{j-1}^n)],$$

and

$$\sum_{j=1}^{J-1}(\delta^2v_j^{n+1},v_j^{n+1}-v_j^n)h=-\frac{1}{2}\|\delta v_h^{n+1}\|_2^2+\frac{1}{2}\|\delta v_h^n\|_2^2-\frac{1}{2}\|\delta v_h^{n+1}-\delta v_h^n\|_2^2.$$

When $0\leqslant\lambda_j\leqslant1$, $0\leqslant\mu_j\leqslant1$, $\lambda_j+\mu_{j+1}\geqslant1$, there is

$$\begin{aligned}&-r\sum_{j=1}^{J-1}((1-\lambda_j)(\Delta_\tau v_{j+1}^{n+1}-\Delta_\tau v_{j+1}^n)+(1-\mu_j)(\Delta_\tau v_{j-1}^{n+1}-\Delta_\tau v_{j-1}^n),v_j^{n+1}-v_j^n)h\\ &-\frac{1}{2}\|\delta v_h^{n+1}-\delta v_h^n\|_2^2\leqslant\frac{\tau^2}{2h}\sum_{j=0}^{J-1}[-(1-\lambda_j)|\Delta_\tau v_{j+1}^{n+1}|^2-(1-\mu_{j+1})|\Delta_\tau v_j^{n+1}|^2\\ &\qquad\qquad+(1-\lambda_j)|\Delta_\tau v_{j+1}^n|^2+(1-\mu_{j+1})|\Delta_\tau v_j^n|^2]\\ &\qquad\qquad=-\frac{\tau r}{2}\sum_{j=1}^{J-1}(2-\lambda_{j-1}-\mu_{j+1})(|\Delta_\tau v_j^{n+1}|^2-|\Delta_\tau v_j^n|^2)h.\end{aligned}$$

The following elementary inequality is used,

$$\begin{aligned}&-\frac{1}{2}|a_{j+1}-a_j|^2-(2-\lambda_j-\mu_{j+1})(a_{j+1},a_j)+(1-\lambda_j)(b_{j+1},a_j)+(1-\mu_{j+1})(b_j,a_{j+1})\\ &\leqslant-\frac{1}{2}(1-\lambda_j)|a_{j+1}|^2-\frac{1}{2}(1-\mu_{j+1})|a_j|^2+\frac{1}{2}(1-\lambda_j)|b_{j+1}|^2+\frac{1}{2}(1-\mu_{j+1})|b_j|^2,\\ &\qquad\forall a_j,a_{j+1},b_j,b_{j+1}\in R^m.\end{aligned}$$

Then there are

$$\|\delta v_h^{n+1}\|_2^2-\|\delta v_h^n\|_2^2+2\tau\sum_{j=1}^{J-1}(\overset{*}{\delta}{}^{2}v_j^{n+1},A_j^{n+1}\,\overset{*}{\delta}{}^{2}v_j^{n+1})$$

$$
\begin{aligned}
&+\tau r\sum_{j=1}^{J-1}(2-\lambda_{j-1}-\mu_{j+1})(|\Delta_\tau v_j^{n+1}|^2-|\Delta_\tau v_j^{n}|^2)h\\
&\leqslant -2\tau\sum_{j=0}^{J-1}(\overset{*}{\delta}{}^2 v_j^{n+1}, f_j^{n+1})h \leqslant 2\tau\Big|\sum_{j=1}^{J-1}(\overset{*}{\delta}{}^2 v_j^{n+1}, f_j^{n+1})h\Big|\\
&\leqslant \frac{\tau}{2}\sum_{j=1}^{J-1}(\overset{*}{\delta}{}^2 v_j^{n+1}, A_j^{n+1}\,\overset{*}{\delta}{}^2 v_j^{n+1})h + 2\tau\sum_{j=1}^{J-1}\frac{|f_j^{n+1}|^2}{\sigma(A_j^{n+1})}h, \qquad (3.2)
\end{aligned}
$$

where the Cauchy inequality is used in the right hand of the above inequality, and we denote

$$\sigma(A)=\inf_{\xi\in R^m}\frac{(\xi,A\xi)}{|\xi|^2}.$$

3.2 Using the equality (2.4), for any $\varepsilon>0$, there is

$$
\begin{aligned}
\varepsilon\sum_{j=1}^{J-1}|\Delta_\tau v_j^{n+1}|^2h &= \varepsilon\sum_{j=1}^{J-1}|A_j^{n+1}\,\overset{*}{\delta}{}^2 v_j^{n+1}+f_j^{n+1}|^2h\\
&\leqslant 2\varepsilon\sum_{j=1}^{J-1}\frac{\rho^2(A_j^{n+1})}{\sigma(A_j^{n+1})}(\overset{*}{\delta}{}^2 v_j^{n+1}, A_j^{n+1}\,\overset{*}{\delta}{}^2 v_j^{n+1})h + 2\varepsilon\sum_{j=1}^{J-1}|f_j^{n+1}|^2h, \qquad (3.3)
\end{aligned}
$$

where

$$\rho(A)=\sup_{\xi\in R^m}\frac{|A\xi|}{|\xi|}.$$

By using the condition **(II)** and the definition (2.7), there is

$$
\begin{aligned}
\sum_{j=1}^{J-1}|f_j^{n+1}|^2h \leqslant C(1&+\|v_h^{n-1}\|_2^2+\|v_h^{n}\|_2^2+\|v_h^{n+1}\|_2^2+\|\delta v_h^{n-1}\|_2^2\\
&+\|\delta v_h^{n}\|_2^2+\|\delta v_h^{n+1}\|_2^2+\tau\Lambda\|\Delta_\tau v_h^{n}\|^2+\tau\Lambda\|\Delta_\tau v_h^{n+1}\|^2). \qquad (3.4)
\end{aligned}
$$

Combining (3.2)–(3.4) and taking $\varepsilon>0$ such that

$$\varepsilon\max_{(x,t)\in Q_T,u\in R^m}\frac{\rho^2(A)}{\sigma(A)}\leqslant\frac{1}{4},$$

we get

$$
\begin{aligned}
&\|\delta v_h^{n+1}\|_2^2-\|\delta v_h^{n}\|_2^2+\tau r\sum_{j=1}^{J-1}(2-\lambda_{j-1}-\mu_{j+1})(|\Delta_\tau v_j^{n+1}|^2-|\Delta_\tau v_j^{n}|^2)h\\
&\quad+2\tau\sum_{j=1}^{J-1}(\overset{*}{\delta}{}^2 v_j^{n+1}, A_j^{n+1}\,\overset{*}{\delta}{}^2 v_j^{n+1})h+\varepsilon\tau\|\Delta_\tau v_h^{n+1}\|_2^2\\
&\leqslant \tau\sum_{j=1}^{J-1}(\overset{*}{\delta}{}^2 v_j^{n+1}, A_j^{n+1}\,\overset{*}{\delta}{}^2 v_j^{n+1})h+2\tau\sum_{j=1}^{J-1}\left(\frac{1}{\sigma(A_j^{n+1})}+\varepsilon\right)|f_j^{n+1}|^2h\\
&\leqslant \tau\sum_{j=1}^{J-1}(\overset{*}{\delta}{}^2 v_j^{n+1}, A_j^{n+1}\,\overset{*}{\delta}{}^2 v_j^{n+1})h\\
&\quad+C\tau(1+\|\delta v_h^{n-1}\|_2^2+\|\delta v_h^{n}\|_2^2+\|\delta v_h^{n+1}\|_2^2+\tau\Lambda\|\Delta_\tau v_h^{n}\|^2+\tau\Lambda\|\Delta_\tau v_h^{n+1}\|^2),
\end{aligned}
$$

where the condition **(I)** and Lemma 2.3 (ii) are used. Taking $C\tau\Lambda \leqslant \frac{\varepsilon}{2}$, we have

$$\begin{aligned}&\|\delta v_h^{n+1}\|_2^2 - \|\delta v_h^n\|_2^2 + \tau r\sum_{j=1}^{J-1}(2-\lambda_{j-1}-\mu_{j+1})(|\Delta_\tau v_j^{n+1}|^2 - |\Delta_\tau v_j^n|^2)h \\ &+\tau\sigma_0\sum_{j=1}^{J-1}|\overset{*}{\delta}{}^2 v_j^{n+1}|^2 h + \frac{\varepsilon\tau}{2}(\|\Delta_\tau v_h^{n+1}\|_2^2 - \|\Delta_\tau v_h^n\|_2^2) \\ &\leqslant C\tau(1+\|\delta v_h^{n-1}\|_2^2+\|\delta v_h^n\|_2^2+\|\delta v_h^{n+1}\|_2^2). \end{aligned} \tag{3.5}$$

3.3 Therefore, for $n \geqslant 1$, there is

$$\begin{aligned}&\|\delta v_h^{n+1}\|_2^2 + \tau r\sum_{j=1}^{J-1}(2-\lambda_{j-1}-\mu_{j+1})|\Delta_\tau v_j^{n+1}|^2 + \tau\sigma_0\sum_{j=1}^{J-1}|\overset{*}{\delta}{}^2 v_j^{n+1}|^2 h + \frac{\varepsilon\tau}{2}\|\Delta_\tau v_h^{n+1}\|_2^2 \\ &\leqslant C\left(1+\sum_{k=0}^{n+1}\|\delta v_h^k\|_2^2\tau + r\tau\|\Delta_\tau v_h^1\|_2^2\right). \end{aligned} \tag{3.6}$$

For $n = 0$ in (2.4), it follows

$$\|\delta v_h^1\|_2^2 + \tau\|\Delta_\tau v_h^1\|_2^2 \leqslant C(1+\|\delta v_h^0\|_2^2).$$

Combining the previous two inequalities and using Lemma 2.2 and Lemma 2.3, we have

$$\max_{1\leqslant n\leqslant N}(\|\delta v_h^n\|_2, \|v_h^n\|_2) \leqslant C(1+\|\delta\varphi_h\|_2), \tag{3.7}$$

$$\left(\sum_{n=0}^{N-1}\|\overset{*}{\delta}{}^2 v_h^{n+1}\|_2^2\tau\right)^{\frac{1}{2}}, \quad \left(\sum_{n=0}^{N-1}\|\Delta_\tau v_h^{n+1}\|_2^2\tau\right)^{\frac{1}{2}} \leqslant C. \tag{3.8}$$

Obviously,

$$\delta^2 v_j^{n+1} = \overset{*}{\delta}{}^2 v_j^{n+1} + r[(1-\lambda_j)(\Delta_\tau v_{j+1}^{n+1} - \Delta_\tau v_{j+1}^n) + (1-\mu_j)(\Delta_\tau v_{j-1}^{n+1} - \Delta_\tau v_{j-1}^n)].$$

By using (3.7)–(3.8) and Lemma 2.3, we have

$$\left(\sum_{n=0}^{N-1}\|\delta^2 v_h^{n+1}\|_2^2\tau\right)^{\frac{1}{2}}, \quad \max_{n=0,1,\dots,N-1}\|v_h^{n+1}\|_\infty \leqslant C. \tag{3.9}$$

3.4 Then we have proved the following result.

Theorem 1. *Suppose that the conditions* **(I)**–**(IV)** *hold, and for any* $1 \leqslant j \leqslant J-1$, *there are* $0 \leqslant \lambda_j \leqslant 1, 0 \leqslant \mu_j \leqslant 1, \lambda_j + \mu_{j+1} \geqslant 1$. *Then there is a positive constant* τ_0 *depending on* Λ *such that, when* $\tau \leqslant \tau_0$, *the solutions of the parallel difference schemes* (2.4)–(2.6) *satisfy the estimates* (3.7)–(3.9).

By using the Leray–Schauder fixed point theorem in a finite dimensional space (see [11]) and the inequalities (3.7)–(3.9), we can get the existence of the solution $v_\Delta = v_h^\tau = \{v_j^n | j = 0,1,\dots,J; n = 0,1,\dots,N\}$ of the parallel schemes (2.4)–(2.6).

Theorem 2. *Suppose that the conditions of Theorem 1 hold. Then there is a positive constant* τ_0 *depending on* Λ, *when* $\tau \leqslant \tau_0$, *at least one solution* $v_\Delta = v_h^\tau = \{v_j^n | j = 0,1,\dots,J; n = 0,1,\dots,N\}$ *of the parallel difference schemes* (2.4)–(2.6) *exists.*

4 Uniqueness

4.1 The values of $\{v_j^{n-1} | j = 0, 1, \dots, J\}$ and $\{v_j^n | j = 0, 1, \dots, J\}$ are given in the schemes (2.4)–(2.6). Let $\{v_j^{n+1} | j = 0, 1, \dots, J\}$ and $\{\overline{v}_j^{n+1} | j = 0, 1, \dots, J\}$ be two solutions of the schemes (2.4)–(2.6) at the time layer $n + 1$. When $n = 0$, it is obvious that $v_j^1 = \overline{v}_j^1, 0 \leqslant j \leqslant J$. Next suppose $n \geqslant 1$, and then there are

$$\begin{aligned} &\frac{v_j^{n+1} - v_j^n}{\tau} = A_j^{n+1} \overset{*}{\delta}{}^{2} v_j^{n+1} + f_j^{n+1}, \quad (j = 1, 2, \dots, J-1) \\ &v_0^{n+1} = v_J^{n+1} = 0; \end{aligned}$$

and

$$\begin{aligned} &\frac{\overline{v}_j^{n+1} - v_j^n}{\tau} = \overline{A}_j^{n+1} \overset{*}{\delta}{}^{2} \overline{v}_j^{n+1} + \overline{f}_j^{n+1}, \quad (j = 1, 2, \dots, J-1) \\ &\overline{v}_0^{n+1} = \overline{v}_J^{n+1} = 0, \end{aligned}$$

where $\overline{A}_j^{n+1}$ and $\overline{f}_j^{n+1}$ $(j = 1, 2, \dots, J-1)$ are obtained by replacing v_j^{n+1} $(j = 0, 1, \dots, J)$ by $\overline{v}_j^{n+1}$ $(j = 0, 1, \dots, J)$ in A_j^{n+1} and f_j^{n+1} $(j = 1, 2, \dots, J-1)$ respectively. Denote $w_j \equiv v_j^{n+1} - \overline{v}_j^{n+1}$. Then

$$w_j = \tau A_j^{n+1} \overset{*}{\delta}{}^{2} w_j + \tau R_j^n, \quad (j = 1, 2, \dots, J-1), \tag{4.1}$$

$$w_0 = w_J = 0, \tag{4.2}$$

where $\overset{*}{\delta}{}^{2} w_j = \overset{*}{\delta}{}^{2} v_j^{n+1} - \overset{*}{\delta}{}^{2} \overline{v}_j^{n+1} = \frac{1}{h^2}(\lambda_j w_{j+1} - 2w_j + \mu_j w_{j-1})$, and

$$R_j^n = (A_j^{n+1} - \overline{A}_j^{n+1}) \overset{*}{\delta}{}^{2} \overline{v}_j^{n+1} + f_j^{n+1} - \overline{f}_j^{n+1}.$$

4.2 Making the scalar product of $\overset{*}{\delta}{}^{2} w_j h\tau$ with (4.1), and summing up the resulting products for $j = 1, 2, \dots, J-1$, and then using the same proof as that of sec. 3, we have

$$\|\delta w_h\|_2^2 + 2\tau \sum_{j=1}^{J-1} (\overset{*}{\delta}{}^{2} w_j, A_j^{n+1} \overset{*}{\delta}{}^{2} w_j) h + \tau \sum_{j=1}^{J-1} (2 - \lambda_{j-1} - \mu_{j+1}) w_j^2 h \leqslant 2\tau \sum_{j=1}^{J-1} |(\overset{*}{\delta}{}^{2} w_j, R_j^n)| h.$$

Then

$$\|\delta w_h\|_2^2 + \tau\sigma_0 \| \overset{*}{\delta}{}^{2} w_h\|_2^2 \leqslant C\tau \sum_{j=1}^{J-1} |R_j^n|^2 h. \tag{4.3}$$

Using the condition **(I)** and the definition (2.1), and using Lemma 2.3 (i), for any $\varepsilon_1 > 0$, there are

$$\begin{aligned} &\|A_h^{n+1} - \overline{A}_h^{n+1}\|_\infty^2 \leqslant \varepsilon_1 \|\delta w_h\|_2^2 + \frac{C}{\varepsilon_1} \|w_h\|_2^2, \\ &\|f_h^{n+1} - \overline{f}_h^{n+1}\|_2^2 \leqslant C\bigg(\|\delta w_h\|_2^2 + \frac{1}{h} \sum_{j=1}^{J-1} (|\lambda_j|^2 w_{j+1}^2 + |\mu_j|^2 w_{j-1}^2) \bigg). \end{aligned}$$

and using the inequality (3.8), we have

$$\tau \| \overset{*}{\delta}{}^{2} \overline{v}_h^{n+1} \|_2^2 \leqslant C.$$

Therefore

$$\tau\sum_{j=1}^{J-1}|R_j^n|^2h \leqslant C(\varepsilon_1+\tau)\|\delta w_h\|_2^2 + C\Lambda\|w_h\|_2^2. \tag{4.4}$$

Substituting (4.4) into (4.3), we get

$$\|\delta w_h\|_2^2 + \tau\sigma_0\|\overset{*}{\delta}{}^2 w_h\|_2^2 \leqslant C(\varepsilon_1+\tau)\|\delta w_h\|_2^2 + C\Lambda\|w_h\|_2^2.$$

Taking $\varepsilon_1 \leqslant \frac{1}{4C}$ and $\tau \leqslant \frac{1}{4C}$, there is

$$\|\delta w_h\|_2^2 + \tau\sigma_0\|\overset{*}{\delta}{}^2 w_h\|_2^2 \leqslant C\|w_h\|_2^2, \tag{4.5}$$

where C is a constant.

4.3 In order to estimate $\|w_h\|_2^2$, making the scalar product of w_jh with (4.1), and summing up the resulting products for $j = 1, 2, \ldots, J-1$, we get

$$\|w_h\|_2^2 = \tau\sum_{j=1}^{J-1}(w_j, A_j^{n+1}\overset{*}{\delta}{}^2 w_j)h + \tau\sum_{j=1}^{J-1}(w_j, R_j^n)h \equiv I + II. \tag{4.6}$$

For any $\varepsilon_2 > 0$ and $\varepsilon_3 > 0$, there are

$$I \leqslant \tau\left(\varepsilon_2\|\overset{*}{\delta}{}^2 w_h\|_2^2 + \frac{C}{\varepsilon_2}\|w_h\|_2^2\right),$$

$$II \leqslant \frac{\tau}{\varepsilon_3}\|w_h\|_2^2 + \varepsilon_3\tau\sum_{j=1}^{J-1}|R_j^n|^2h \leqslant C\varepsilon_3(1+\tau)\|\delta w_h\|_2^2 + \left(\varepsilon_3 C\Lambda + \frac{\tau}{\varepsilon_3}\right)\|w_h\|_2^2.$$

Substituting the above two inequalities into (4.6), we have

$$\|w_h\|_2^2 \leqslant 2\left(C\varepsilon_3\Lambda + \frac{\tau}{\varepsilon_3} + \frac{C\tau}{\varepsilon_2}\right)\|w_h\|_2^2 + C\varepsilon_3(1+\tau)\|\delta w_h\|_2^2 + \tau\varepsilon_2\|\overset{*}{\delta}{}^2 w_h\|_2^2. \tag{4.7}$$

4.4 Combining (4.5) and (4.7), and taking $\varepsilon_2 = \frac{\sigma_0}{2C}$ and $\varepsilon_3 = \min(\frac{1}{8C\Lambda}, \frac{1}{2C})$, we get, when $\tau \leqslant \min(\frac{\varepsilon_2}{8C}, \frac{\varepsilon_3}{8}, 1)$,

$$\|\delta w_h\|_2^2 + \tau\sigma_0\|\overset{*}{\delta}{}^2 w_h\|_2^2 \leqslant 0.$$

Therefore $w_h \equiv 0$, i.e. $v_h^{n+1} = \bar{v}_h^{n+1}$. So we have proved the following theorem of the uniqueness of the parallel schemes (2.4)–(2.6).

Theorem 3. *Suppose that the conditions of Theorem* 1 *hold. Then there is a positive constant* τ_0 *depending on* Λ*, when* $\tau \leqslant \tau_0$*, the solution* $v_\Delta = v_h^\tau = \{v_j^n | j = 0, 1, \ldots, J; n = 0, 1, \ldots, N\}$ *of the parallel scheme* (2.4)–(2.6) *is unique.*

5 Stability

5.1 Let us now consider the unconditional stability of the difference schemes (2.4)–(2.6). Suppose that $\tilde{v}_\Delta = \{\tilde{v}_j^n | j = 0, 1, \ldots, J; n = 0, 1, \ldots, N\}$ satisfies

$$\frac{\tilde{v}_j^{n+1} - \tilde{v}_j^n}{\tau} = \tilde{A}_j^{n+1}\overset{*}{\delta}{}^2\tilde{v}_j^{n+1} + \tilde{f}_j^{n+1}, \quad (j = 1, 2, \ldots, J-1; n = 0, 1, \ldots, N-1), \tag{5.1}$$

$$\tilde{v}_0^{n+1} = \tilde{v}_J^{n+1} = 0, \quad (n = 0, 1, \ldots, N-1), \tag{5.2}$$

$$\tilde{v}_j^0 = \tilde{\varphi}_j, \quad (j = 0, 1, \dots, J), \tag{5.3}$$

where

$$\tilde{A}_j^{n+1} = \tilde{A}(x_j, t^{n+1}, \bar{\delta}^0 v_j^{n+1}), \qquad \tilde{f}_j^{n+1} = \tilde{f}(x_j, t^{n+1}, \hat{\delta}^0 v_j^{n+1}, \bar{\delta}^1 v_j^{n+1}).$$

Suppose that the $m \times m$ coefficient matrix $\tilde{A}(x,t,u)$, m-dimensional vector function $\tilde{f}(x,t,u,p)$ and $\tilde{\varphi}(x)$ approximate $A(x,t,u)$, $f(x,t,u,p)$ and $\varphi(x)$ respectively, and satisfy the conditions **(I)–(III)**. Then $w_\Delta = v_\Delta - \tilde{v}_\Delta = \{w_j^n = v_j^n - \tilde{v}_j^n. \, | j = 0, 1, \dots, J; n = 0, 1, \dots, N\}$ satisfies

$$\frac{w_j^{n+1} - w_j^n}{\tau} = A_j^{n+1} \overset{*}{\delta}{}^{2} w_j^{n+1} + (A_j^{n+1} - \tilde{A}_j^{n+1}) \overset{*}{\delta}{}^{2} \tilde{v}_j^{n+1} + (f_j^{n+1} - \tilde{f}_j^{n+1}), \tag{5.4}$$

$$(j = 1, 2, \dots, J-1; n = 0, 1, \dots, N-1),$$

$$w_0^{n+1} = w_J^{n+1} = 0, \ \ (n = 0, 1, \dots, N-1), \tag{5.5}$$

$$w_j^0 = \varphi_j - \tilde{\varphi}_j, \quad (j = 0, 1, \dots, J). \tag{5.6}$$

The equality (5.4) can be rewritten as

$$\frac{w_j^{n+1} - w_j^n}{\tau} = A_j^{n+1} \overset{*}{\delta}{}^{2} w_j^{n+1} + S_j^{n+1} + R_j^{n+1}, \tag{5.7}$$

where

$$\begin{aligned} S_j^{n+1} &= B(v,\tilde{v})_j^{n+1} \bar{\delta}^0 w_j^{n+1} + C(v,\tilde{v})_j^{n+1} \hat{\delta}^0 w_j^{n+1} + D(v,\tilde{v})_j^{n+1} \bar{\delta}^1 w_j^{n+1}, \\ R_j^{n+1} &= A[\tilde{v}]_j^{n+1} \overset{*}{\delta}{}^{2} \tilde{v}_j^{n+1} + F[\tilde{v}]_j^{n+1}, \\ B(v,\tilde{v})_j^{n+1} &= (\tilde{A}_u)_j^{n+1} \overset{*}{\delta}{}^{2} \tilde{v}_j^{n+1}, \ \ C(v,\tilde{v})_j^{n+1} = (\tilde{f}_u)_j^{n+1}, \ \ D(v,\tilde{v})_j^{n+1} = (\tilde{f}_p)_j^{n+1}, \end{aligned}$$

and for $\bar{\delta}^0 w_j^{n+1} \neq 0$, we define

$$(\tilde{A}_u)_j^{n+1} = \frac{A(x_j, t^{n+1}, \bar{\delta}^0 v_j^{n+1}) - A(x_j, t^{n+1}, \bar{\delta}^0 \tilde{v}_j^{n+1})}{\bar{\delta}^0 w_j^{n+1}};$$

and for $\bar{\delta}^0 w_j^{n+1} = 0$, we define $(\tilde{A}_u)_j^{n+1} = 0$. And for $\hat{\delta}^0 w_j^{n+1} \neq 0$, define

$$(\tilde{f}_u)_j^{n+1} = \frac{f(x_j, t^{n+1}, \hat{\delta}^0 v_j^{n+1}, \bar{\delta}^1 \tilde{v}_j^{n+1}) - f(x_j, t^{n+1}, \hat{\delta}^0 \tilde{v}_j^{n+1}, \bar{\delta}^1 \tilde{v}_j^{n+1})}{\hat{\delta}^0 w_j^{n+1}},$$

and for $\hat{\delta}^0 w_j^{n+1} = 0$, define $(\tilde{f}_u)_j^{n+1} = 0$. For $(\tilde{f}_p)_j^{n+1}$ there are similar expressions. Denote

$$\begin{aligned} A[\tilde{v}]_j^{n+1} &= A(x_j, t^{n+1}, \bar{\delta}^0 \tilde{v}_j^{n+1}) - \tilde{A}(x_j, t^{n+1}, \bar{\delta}^0 \tilde{v}_j^{n+1}), \\ F[\tilde{v}]_j^{n+1} &= f(x_j, t^{n+1}, \hat{\delta}^0 \tilde{v}_j^{n+1}, \bar{\delta}^1 \tilde{v}_j^{n+1}) - \tilde{f}(x_j, t^{n+1}, \hat{\delta}^0 \tilde{v}_j^{n+1}, \bar{\delta}^1 \tilde{v}_j^{n+1}). \end{aligned}$$

5.2 Making the scalar product of $\overset{*}{\delta}{}^{2} w_j^{n+1} h\tau$ with (5.7), summing up the resulting products for $j = 1, 2, \dots, J-1$, and using the same proof as that in sec. 3, we get

$$\begin{aligned} &\|\delta w_h^{n+1}\|_2^2 - \|\delta w_h^n\|_2^2 + 2\tau \sum_{j=1}^{J-1} (\overset{*}{\delta}{}^{2} w_j^{n+1}, A_j^{n+1} \overset{*}{\delta}{}^{2} w_j^{n+1}) h \\ &\qquad + \tau r \sum_{j=1}^{J-1} (2 - \lambda_{j-1} - \mu_{j+1})(|\Delta_\tau v_j^{n+1}|^2 - |\Delta_\tau v_j^n|^2) h \end{aligned}$$

$$\leqslant 2\tau\left|\sum_{j=1}^{J-1}(\overset{*}{\delta}{}^2 w_j^{n+1}, S_j^{n+1}+R_j^{n+1})h\right|.$$

Therefore it follows

$$\|\delta w_h^{n+1}\|_2^2-\|\delta w_h^n\|_2^2+\tau r\sum_{j=1}^{J-1}(2-\lambda_{j-1}-\mu_{j+1})(|\Delta_\tau w_j^{n+1}|^2-|\Delta_\tau w_j^n|^2)h$$

$$+\sigma_0\tau\|\overset{*}{\delta}{}^2 w_h^{n+1}\|_2^2\leqslant\frac{4\tau}{\sigma_0}\sum_{j=1}^{J-1}(|S_j^{n+1}|^2+|R_j^{n+1}|^2)h. \tag{5.8}$$

By using the conditions **(I)**–**(II)**, the definition (2.7) and Lemma 2.3, we have

$$\sum_{j=1}^{J-1}|B_j^{n+1}\bar{\delta}^0 w_j^{n+1}|^2h\leqslant C(\|w_h^{n-1}\|_\infty^2+\|w_h^n\|_\infty^2+\|w_h^{n+1}\|_\infty^2)\|\overset{*}{\delta}{}^2\tilde{v}_h^{n+1}\|_2^2,$$

$$\sum_{j=1}^{J-1}|C_j^{n+1}\hat{\delta}^0 w_j^{n+1}|^2h\leqslant C(\|w_h^{n-1}\|_2^2+\|w_h^n\|_2^2+\|w_h^{n+1}\|_2^2),$$

$$\sum_{j=1}^{J-1}|D_j^{n+1}\bar{\delta}^1 w_j^{n+1}|^2h\leqslant C(\|\delta w_h^{n-1}\|_2^2+\|\delta w_h^n\|_2^2+\|\delta w_h^{n+1}\|_2^2$$
$$+\Lambda\tau\|\Delta_\tau w_h^{n+1}\|_2^2+\Lambda\tau\|\Delta_\tau w_h^n\|_2^2),$$

$$\sum_{j=1}^{J-1}|R_j^{n+1}|^2h\leqslant C(\|A[\tilde{v}]_h^{n+1}\|_\infty\|\overset{*}{\delta}{}^2\tilde{v}_h^{n+1}\|_2^2+\|F[\tilde{v}]_h^{n+1}\|_2^2),$$

where C is a constant independent of h and τ.

Substituting the above inequalities into (5.8), we get

$$\begin{aligned}
&\|\delta w_h^{n+1}\|_2^2-\|\delta w_h^n\|_2^2+\tau r\sum_{j=1}^{J-1}(2-\lambda_{j-1}-\mu_{j+1})(|\Delta_\tau w_j^{n+1}|^2-|\Delta_\tau w_j^n|^2)h+\sigma_0\tau\|\overset{*}{\delta}{}^2 w_h^{n+1}\|_2^2\\
&\leqslant C\tau\{\|\delta w_h^{n-1}\|_2^2+\|\delta w_h^n\|_2^2+\|\delta w_h^{n+1}\|_2^2\\
&\qquad+(\|w_h^{n-1}\|_\infty^2+\|w_h^n\|_\infty^2+\|w_h^{n+1}\|_\infty^2)\|\overset{*}{\delta}{}^2\tilde{v}_h^{n+1}\|_2^2+\Lambda\tau\|\Delta_\tau w_h^{n+1}\|_2^2\\
&\qquad+\Lambda\tau\|\Delta_\tau w_h^n\|_2^2+\|A[\tilde{v}]_h^{n+1}\|_\infty^2\|\overset{*}{\delta}{}^2\tilde{v}_h^{n+1}\|_2^2+\|F[\tilde{v}]_h^{n+1}\|_2^2\}.
\end{aligned} \tag{5.9}$$

5.3 When $n\geqslant 1$, from the equality (5.7), there is

$$\begin{aligned}
\tau\|\Delta_\tau w_h^{n+1}\|_2^2\leqslant& C\tau\{\|\overset{*}{\delta}{}^2 w_h^{n+1}\|_2^2+(\|w_h^{n-1}\|_\infty^2+\|w_h^n\|_\infty^2+\|w_h^{n+1}\|_\infty^2)\|\overset{*}{\delta}{}^2\tilde{v}_h^{n+1}\|_2^2\\
&+\|\delta w_h^{n-1}\|_2^2+\|\delta w_h^n\|_2^2+\|\delta w_h^{n+1}\|_2^2+\Lambda\tau\|\Delta_\tau w_h^n\|_2^2+\Lambda\tau\|\Delta_\tau w_h^{n+1}\|_2^2\\
&+\|A[\tilde{v}]_h^{n+1}\|_\infty^2\|\overset{*}{\delta}{}^2\tilde{v}_h^{n+1}\|_2^2+\|F[\tilde{v}]_h^{n+1}\|_2^2\}.
\end{aligned}$$

Taking $\tau>0$ such that $C\Lambda\tau\leqslant\frac{1}{2}$. Then

$$\sum_{k=0}^{n}\|\Delta_\tau w_h^{k+1}\|_2^2\tau\leqslant C\bigg(\sum_{k=0}^{n}\|\overset{*}{\delta}{}^2 w_h^{n+1}\|_2^2\tau+\max_{0\leqslant k\leqslant n+1}\|w_h^k\|_\infty^2+\sum_{k=0}^{n+1}\|\delta w_h^k\|_2^2\tau\bigg). \tag{5.10}$$

5.4 Using the inequality (5.9), we have

$$\|\delta w_h^{n+1}\|_2^2+\sigma_0\sum_{k=0}^{n}\|\overset{*}{\delta}{}^2 w_h^{k+1}\|_2^2\tau+\tau r\sum_{j=1}^{J-1}(2-\lambda_{j-1}-\mu_{j+1})|\Delta_\tau w_j^{n+1}|^2h$$

$$\leqslant C\Big(\sum_{k=0}^{n}\|\delta w_h^{k+1}\|_2^2\tau+\max_{0\leqslant k\leqslant n+1}\|w_h^k\|_\infty^2+I_0\Big)+C\tau\sum_{k=0}^{n}\|\Delta_\tau w_h^{k+1}\|_2^2\tau,$$

where

$$I_0\equiv\|\delta w_h^0\|_2^2+\max_{0\leqslant n\leqslant N-1}\|A[\tilde v]_h^{n+1}\|_\infty^2+\sum_{n=0}^{N-1}\|F[\tilde v]_h^{n+1}\|_2^2\tau.$$

Using the inequality (5.10) and taking $C\varepsilon<\frac{\sigma_0}{4}$, we get

$$\|\delta w_h^{n+1}\|_2^2+\frac{3\sigma_0}{4}\sum_{k=0}^{n}\|\overset{*}{\delta}{}^2 w_h^{k+1}\|_2^2\tau+\tau r\sum_{j=1}^{J-1}(2-\lambda_{j-1}-\mu_{j+1})|\Delta_\tau w_j^{n+1}|^2h+\varepsilon\sum_{k=0}^{n}\|\Delta_\tau w_h^{k+1}\|_2^2\tau$$

$$\leqslant C\Big(\sum_{k=0}^{n}\|\delta w_h^{k+1}\|_2^2\tau+\max_{0\leqslant k\leqslant n+1}\|w_h^k\|_\infty^2+I_0\Big)+C\tau\sum_{k=0}^{n}\|\Delta_\tau w_h^{k+1}\|_2^2\tau. \tag{5.11}$$

Substituting the inequality (5.10) into the right hand of the inequality (5.11) and taking $C\tau<\frac{\sigma_0}{4}$, we have

$$\|\delta w_h^{n+1}\|_2^2+\frac{\sigma_0}{2}\sum_{k=0}^{n}\|\overset{*}{\delta}{}^2 w_h^{k+1}\|_2^2\tau+\tau r\sum_{j=1}^{J-1}(2-\lambda_{j-1}-\mu_{j+1})|\Delta_\tau w_j^{n+1}|^2h+\varepsilon\sum_{k=0}^{n}\|\Delta w_h^{k+1}\|_2^2\tau$$

$$\leqslant C\Big(\sum_{k=0}^{n}\|\delta w_h^{k+1}\|_2^2\tau+\max_{0\leqslant k\leqslant n+1}\|w_h^k\|_\infty^2+I_0\Big).$$

5.5 By using the above inequality and Lemma 2.2, for $0\leqslant n\leqslant N-1$, there holds

$$\|\delta w_h^{n+1}\|_2^2+\frac{\sigma_0}{2}\sum_{k=0}^{n}\|\overset{*}{\delta}{}^2 w_h^{k+1}\|_2^2\tau+\tau r\sum_{j=1}^{J-1}(2-\lambda_{j-1}-\mu_{j+1})|\Delta_\tau w_j^{n+1}|^2h$$

$$+\varepsilon\sum_{k=0}^{n}\|\Delta_\tau w_h^{k+1}\|_2^2\tau\leqslant C\Big(\max_{0\leqslant k\leqslant n+1}\|w_h^k\|_\infty^2+I_0\Big), \tag{5.12}$$

which gives, by Lemma 2.3, for any $\varepsilon_4>0$

$$\|\delta w_h^{n+1}\|_2^2+\frac{\sigma_0}{2}\sum_{k=0}^{n}\|\overset{*}{\delta}{}^2 w_h^{k+1}\|_2^2\tau+\varepsilon\sum_{k=0}^{n}\|\Delta_\tau w_h^{k+1}\|_2^2\tau$$

$$\leqslant C\Big(\varepsilon_4\max_{0\leqslant k\leqslant n+1}\|\delta w_h^k\|_2^2+\frac{1}{\varepsilon_4}\max_{0\leqslant k\leqslant n+1}\|w_h^k\|_2^2+I_0\Big). \tag{5.13}$$

5.6 The following lemmas obviously hold.

Lemma 5.1. *If the discrete functions $D(k)$ and $G(k)$ $(0\leqslant k\leqslant N)$ satisfy $G(k)\leqslant G(k+1)$ $(0\leqslant k\leqslant N-1)$, and*

$$D(n)\leqslant\frac{1}{2}\max_{0\leqslant k\leqslant n}D(k)+G(n),\quad n=0,1,\ldots,N,$$

then $D(n) \leqslant 2G(n)$, $n = 0, 1, \dots, N$.

Lemma 5.2. *For any* $\varepsilon_5 > 0$, $n = 0, 1, \dots, N-1$, *there is*

$$\max_{0 \leqslant k \leqslant n} \|w_h^{k+1}\|_2^2 \leqslant \varepsilon_5 \sum_{k=0}^{n} \|\Delta_\tau w_h^{k+1}\|_2^2 \tau + \frac{1}{\varepsilon_5} \sum_{k=0}^{n+1} \|w_h^k\|_2^2 \tau + \|w_h^0\|_2^2.$$

5.7 Taking $C\varepsilon_4 \leqslant \frac{1}{2}$ in the inequality (5.13) and using Lemma 5.1, we have

$$\|\delta w_h^{n+1}\|_2^2 + \frac{\sigma_0}{2} \sum_{k=0}^{n} \|\overset{*}{\delta}{}^2 w_h^{k+1}\|_2^2 \tau + \varepsilon \sum_{k=0}^{n} \|\Delta_\tau w_h^{k+1}\|_2^2 \tau \leqslant C\Big(\max_{0 \leqslant k \leqslant n+1} \|w_h^k\|_2^2 + I_0\Big).$$

By using Lemma 5.2 and letting $C\varepsilon_5 \leqslant \frac{\varepsilon}{2}$, for any $0 \leqslant n \leqslant N-1$, there is

$$\|\delta w_h^{n+1}\|_2^2 + \frac{\sigma_0}{2} \sum_{k=0}^{n} \|\overset{*}{\delta}{}^2 w_h^{k+1}\|_2^2 \tau + \frac{\varepsilon}{2} \sum_{k=0}^{n} \|\Delta_\tau w_h^{k+1}\|_2^2 \tau \leqslant C\Big(\sum_{k=0}^{n+1} \|w_h^k\|_2^2 \tau + I_0\Big).$$

Therewith, by using Lemma 2.2 (ii), for $n = 0, 1, \dots, N-1$, there is

$$\|\delta w_h^{n+1}\|_2^2 + \sum_{k=0}^{n} \|\overset{*}{\delta}{}^2 w_h^{k+1}\|_2^2 \tau + \sum_{k=0}^{n} \|\Delta_\tau w_h^{k+1}\|_2^2 \tau \leqslant C I_0.$$

5.8 It demonstrates that the solution v_Δ of the parallel schemes (2.4)–(2.6) continuously depend on the discrete H^1 norm of the discrete initial value function $\varphi(x)$, the discrete L^∞ norm of the coefficient matrix $A(x, t, u)$ and the discrete L^2 norm of the nonlinear function $f(x, t, u, p)$ in the discrete functional space $W_2^{(2,1)}(Q_\Delta)$. Then we have proved the following stability theorem.

Theorem 4. *Under the conditions of Theorem* 1, *for* $w_\Delta = v_\Delta - \tilde{v}_\Delta = \{w_j^n = v_j^n - \tilde{v}_j^n \,|\, j = 0, 1, \dots, J; n = 0, 1, \dots, N\}$, *there is the following inequality*:

$$\begin{aligned}
\|v_\Delta - \tilde{v}_\Delta\|_{W_2^{(2,1)}(Q_\Delta)}^2 \leqslant K\{&\|\varphi_h - \tilde{\varphi}_h\|_{H_h^1}^2 \\
&+ \max_{|u| \leqslant C} \max_{1 \leqslant j \leqslant J-1, 0 \leqslant n \leqslant N-1} |A(x_j, t^{n+1}, u) - \tilde{A}(x_j, t^{n+1}, u)|^2 \\
&+ \max_{|u| \leqslant C, |p| < \infty} \sum_{j=1}^{J-1} \sum_{n=0}^{N-1} |f(x_j, t^{n+1}, u, p) - \tilde{f}(x_j, t^{n+1}, u, p)|^2 h\tau\},
\end{aligned}$$

where K *is a constant independent of* h *and* τ, *and the constant* C *is determined by the inequality* (3.9), *and*

$$\|\varphi_h\|_{H_h^1}^2 = \|\varphi_h\|_2^2 + \|\delta\varphi_h\|_2^2,$$

$$\|w_\Delta\|_{W_2^{(2,1)}(Q_\Delta)}^2 \equiv \max_{n=0,1,\dots,N} \|w_h^n\|_{H_h^1}^2 + \sum_{n=0}^{N-1} \|\overset{*}{\delta}{}^2 w_h^{n+1}\|_2^2 \tau + \sum_{n=0}^{N-1} \|\Delta_\tau w_h^{n+1}\|_2^2 \tau.$$

6 Convergence

6.1 We will prove the convergence of the solution $v_\Delta = v_h^\tau = \{v_j^n \,|\, j = 0, 1, \dots, J; n = 0, 1, \dots, N\}$ of the difference schemes (2.4)–(2.6) using the inequalities obtained in the previous sections.

Lemma 6.1. *For the discrete solution of* (2.4)–(2.6), *there are*

$$\max_{n=0,1,\dots,N} |v_j^n - v_{j'}^n| \leqslant C|x_j - x_{j'}|^{\frac{1}{2}}, \;\; 0 \leqslant j, j' \leqslant J; \tag{6.1}$$

$$\max_{j=0,1,\dots,J} |v_j^n - v_j^{n'}| \leqslant C|t^n - t^{n'}|^{\frac{1}{4}}, \;\; 0 \leqslant n, n' \leqslant N; \tag{6.2}$$

$$\Big(\sum_{n=0}^{N-1} \sum_{j=0}^{J-m-1} |\delta v_{j+m}^n - \delta v_j^n|^2 h\tau\Big)^{\frac{1}{2}} \leqslant Cmh, \tag{6.3}$$

$$\Big(\sum_{n=0}^{N-s} \sum_{j=0}^{J-1} |\delta v_j^{n+s} - \delta v_j^n|^2 h\tau\Big)^{\frac{1}{2}} \leqslant C(s\tau)^{\frac{1}{4}}, \tag{6.4}$$

where m *and* s *are the integers satisfying* $0 \leqslant m \leqslant J-1, 0 \leqslant s \leqslant N$, *and the constant* C *independent of* h, τ, m, s *and* v_h^τ.

Proof. Obviously

$$|v_j^n - v_{j'}^n| \leqslant \|\delta v_h^n\|_2 |x_j - x_{j'}|^{\frac{1}{2}},$$

and the estimate (6.1) is obtained by using the inequality (3.7). From Lemma 2.3 and the inequalities (3.7)–(3.8), it follows

$$\|v_h^n - v_h^{n'}\|_\infty \leqslant \|\delta(v_h^n - v_h^{n'})\|_2^{\frac{1}{2}} \|v_h^n - v_h^{n'}\|_2^{\frac{1}{2}} \leqslant C\|v_h^n - v_h^{n'}\|_2^{\frac{1}{2}},$$

and

$$\|v_h^n - v_h^{n'}\|_2 \leqslant \Big(\sum_{k=0}^{N-1} \|\Delta_\tau v_h^{k+1}\|_2^2 \tau\Big)^{\frac{1}{2}} |t^n - t^{n'}|^{\frac{1}{2}} \leqslant C|t^n - t^{n'}|^{\frac{1}{2}}.$$

Therefore, the inequality (6.2) has been obtained. Similarly we can get the inequality (6.3) by the inequality (3.9). Notice

$$\|\delta v_h^{n+s} - \delta v_h^n\|_2^2 \leqslant C\|\delta^2 v_h^{n+s} - \delta^2 v_h^n\|_2 \|v_h^{n+s} - v_h^n\|_2 \leqslant C(s\tau)^{\frac{1}{2}},$$

i.e. the inequality (6.4) holds. The proof of Lemma 6.1 is completed.

6.2 For $(x,t) \in Q_j^n$ $(j = 0, 1, \dots, J-1; n = 0, 1, \dots, N-1)$, define the piecewise constant functions as follows:

$$v_h^\tau(x,t) = v_j^{n+1}, \quad v_{xh}^\tau(x,t) = \delta v_j^{n+1} = \frac{1}{2}(\delta v_{j+\frac{1}{2}}^{n+1} + v_{j-\frac{1}{2}}^{n+1}), \;\; v_{th}^\tau(x,t) = \Delta_\tau v_j^{n+1},$$

and define

$$v_{xxh}^\tau(x,t) = \overset{*}{\delta}{}^2 v_j^{n+1}, \;\; (x,t) \in Q_j^n, \;\; (j = 1, 2, \dots, J-1; n = 0, 1, \dots, N-1);$$
$$v_{xxh}^\tau(x,t) = \overset{*}{\delta}{}^2 v_1^{n+1}, \;\; (x,t) \in Q_0^n, \;\; (n = 0, 1, \dots, N-1).$$

Denote that J_0 is the number of the subscript j satisfying $\lambda_j \neq 1$ or $\mu_j \neq 1$, i.e. the number of grids not using the fully implicit scheme. The following condition illustrates that the number of grids not using the fully implicit scheme is not beyond a fixed constant, i.e., the number of sub-domains obtained is uniformly bounded for any $h > 0$ and $\tau > 0$.

(V) For any fixed constant $C > 0$, there holds $J_0 \leqslant C$ for any $h > 0$ and $\tau > 0$.

Lemma 6.2. *Assume that the condition* **(V)** *and the conditions of Theorem 1 hold. Then, when* $h \to 0, \tau \to 0$ *(for some subsequences), there is a function* $u(x,t) \in W_2^{(2,1)}(Q_T)$ *such that*

(i) $v_h^{\tau}(x,t) \to u(x,t)$ *uniformly in* Q_T;

(ii) $v_{xh}^{\tau}(x,t) \to u_x(x,t)$ *strongly in* $L^2(Q_T)$, *and a.e. in* Q_T;

(iii) $v_{xxh}^{\tau}(x,t) \to u_{xx}(x,t)$ *weakly in* $L^2(Q_T)$;

(iv) $v_{th}^{\tau}(x,t) \to u_t(x,t)$ *weakly in* $L^2(Q_T)$.

Proof. Using Lemma 6.1 and the discrete compactness method in [11], we can prove (i) and (ii) by the bilinear interpolation function constructed by $v_h^{\tau}(x,t)$.

Using the estimate (3.8) and the discriminant rule of the compactness set, there is $u''(x,t) \in L^2(Q_T)$ such that $v_{xxh}^{\tau}(x,t) \to u''(x,t)$ weakly in $L^2(Q_T)$. Next we prove that $u''(x,t) = u_{xx}(x,t)$ holds in Q_T. Let $\Phi(x,t) \in C_0^{\infty}(Q_T)$, denote $\Phi_j^n = \Phi(x_j, t^n)$. Define the piecewise constant functions $\Phi_h^{\tau}(x,t) = \Phi_j^{n+1}, \Phi_{xxh}^{\tau}(x,t) = \delta^2\Phi_j^{n+1}$, $(x,t) \in Q_j^n$. When h and τ are small enough, there is

$$
\begin{aligned}
\int\int_{Q_T} & v_{xxh}{}^{\tau}(x,t)\Phi_h^{\tau}(x,t)dxdt \\
&= \sum_{n=0}^{N-1}\sum_{j=1}^{J-1} \overset{*}{\delta}{}^{2} v_j^{n+1}\Phi_j^{n+1}h\tau \\
&= \sum_{n=0}^{N-1}\sum_{j=1}^{J-1} \delta^2 v_j^{n+1}\Phi_j^{n+1}h\tau \\
&\quad - r\sum_{n=0}^{N-1}\sum_{j=1}^{J-1} \frac{(1-\lambda_j)(\Delta_\tau v_{j+1}^{n+1} - \Delta_\tau v_{j+1}^{n}) + (1-\mu_j)(\Delta_\tau v_{j-1}^{n+1} - \Delta_\tau v_{j-1}^{n})}{h}\Phi_j^{n+1}h\tau \\
&\equiv I + II.
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
& I \to \int\int_{Q_T} u(x,t)\Phi_{xx}(x,t)dxdt, \quad h \to 0, \tau \to 0; \\
& |II| \leqslant 4\Lambda \max_{0 \leqslant n \leqslant N-1} \|\Phi_h^{n+1}\|_{\infty}\left(\sum_{n=0}^{N-1} \|\Delta_\tau v_h^{n+1}\|_2^2\tau\right)^{\frac{1}{2}}(TJ_0h)^{\frac{1}{2}},
\end{aligned}
$$

which concludes $II \to 0$ as $h \to 0$ and $\tau \to 0$. Then (iii) holds. Similarly (iv) holds. The proof of Lemma 6.2 is completed.

6.3 For $(x,t) \in Q_j^n$ $(j = 0, 1, \ldots, J-1; n = 0, 1, \ldots, N-1)$, define the piecewise constant functions

$$
\begin{aligned}
& \bar{v}_h^{\tau}(x,t) = \bar{\delta}^0 v_j^{n+1}, \ \hat{v}_h^{\tau}(x,t) = \hat{\delta}^0 v_j^{n+1}, \quad \bar{v}_{xh}^{\tau}(x,t) = \bar{\delta}^1 v_j^{n+1}, \\
& A_h^{\tau}(x,t) = A_j^{n+1}, \quad f_h^{\tau}(x,t) = f_j^{n+1}.
\end{aligned}
$$

Lemma 6.3. *Assume that the conditions of Theorem* 1 *hold. When* $h \to 0, \tau \to 0$ *(for some subsequences), there are*

(i) $\bar{v}_h^{\tau}(x,t) \to u(x,t)$ *and* $\hat{v}_h^{\tau}(x,t) \to u(x,t)$ *strongly in* $L^2(Q_T)$ *and a.e. in* Q_T;

(ii) $\bar{v}_{xh}^{\tau}(x,t) \to u_x(x,t)$ *strongly in* $L^2(Q_T)$ *and a.e. in* Q_T;

(iii) $A_h^\tau(x,t) \to A(x,t,u(x,t))$ *and* Q_T $f_h^\tau(x,t) \to f(x,t,u(x,t),u_x(x,t))$ *strongly in* $L^2(Q_T)$ *and a.e. in* Q_T.

Proof. According to the definition (2.7) and the estimates (3.7)–(3.8), it follows that

$$\begin{aligned}\|\bar{v}_h^\tau(x,t) - v_h^\tau(x,t)\|^2_{L^2(Q_T)} &\leqslant Ch^2 \sum_{n=0}^{N} \|\delta v_h^n\|_2^2 \tau + C\tau^2 \sum_{n=0}^{N-1} \|\Delta_\tau v_h^{n+1}\|_2^2 \tau \\ &\leqslant C(h^2+\tau^2), \\ \int_0^T \|\bar{v}_{xh}^\tau(x,t) - v_{xh}^\tau(x,t)\|_2^2 dt &\leqslant C\left(\frac{\tau}{h}\right)^2 \sum_{j=1}^{J-1}\sum_{n=0}^{N-1} |\Delta_\tau v_j^{n+1}|^2 h\tau + Ch^2 \sum_{j=1}^{J-1}\sum_{n=0}^{N-1} |\delta^2 v_j^{n+1}|^2 h\tau \\ &\leqslant C(\Lambda\tau + h^2).\end{aligned}$$

Using Lemma 6.2 and the above two inequalities, we can get (i) and (ii). Now we prove (iii). By the condition **(I)**, we know that $A(x,t,u) \in C(Q_T \times R^m)$, and using (3.7), there is $\max_{0\leqslant n\leqslant N-1} \|\bar{\delta}^0 v_h^{n+1}\|_\infty \leqslant C$. So we have

$$\begin{aligned}&\|A_h^\tau(x,t) - A(x,t,u(x,t))\|^2_{L^2(Q_T)} \\ &\quad= \sum_{n=0}^{N-1}\sum_{j=0}^{J-1} \int\int_{Q_j^n} |A(x_j,t^{n+1},\bar{\delta}^0 v_j^{n+1}) - A(x,t,u(x,t))|^2 dxdt \\ &\quad\leqslant 2lT \max_{|x'-x|\leqslant h, |t'-t|\leqslant\tau, |v|\leqslant C} |A(x',t',v) - A(x,t,v)|^2 \\ &\qquad + 2C\|\bar{v}_h^\tau(x,t) - u(x,t)\|^2_{L^2(Q_T)},\end{aligned}$$

Hence $A_h^\tau \to A$ strongly in $L^2(Q_T)$. Similarly we can prove that $f_h^\tau \to f$ strongly in $L^2(Q_T)$. The proof of Lemma 6.3 is completed.

6.4 Let $\Phi(x,t) \in C^\infty(Q_T)$ and $\Phi(x,t) = 0$ near $x=0$ and $x=l$. Denote $\Phi_j^n = \Phi(x_j,t^n)$. Define the piecewise constant functions $\Phi_h^\tau(x,t) = \Phi_j^{n+1}$, for $(x,t) \in Q_j^n$. There is

$$\begin{aligned}&\int\int_{Q_T} [v_{th}^\tau(x,t) - A_h^\tau(x,t) v_{xxh}^\tau - f_h^\tau(x,t)]\Phi_h^\tau(x,t) dxdt \\ &\quad= \sum_{n=0}^{N-1}\sum_{j=1}^{J-1} \left[\frac{v_j^{n+1} - v_j^n}{\tau} - A_j^{n+1} \overset{*}{\delta}{}^2 v_j^{n+1} - f_j^{n+1}\right]\Phi_j^{n+1} h\tau = 0.\end{aligned}$$

Let $h \to 0, \tau \to 0$ (for some subsequences). Then

$$\int\int_{Q_T} [u_t(x,t) - A(x,t,u)u_{xx}(x,t) - f(x,t,u)]\Phi(x,t) dxdt = 0.$$

Since the subsequence $v_h^\tau(x,t)$ is uniformly convergent to $u(x,t)$ in Q_T, $u(x,t)$ satisfies the homogeneous boundary conditions (2.2) and the initial condition (2.3). This means that $u(x,t) \in W_2^{(2,1)}(Q_T)$ is just a generalized solution of the problems (2.1)–(2.3), The uniqueness of the generalized solution for the problems (2.1)–(2.3) can be justified by the well-known method (see [11]). Then we obtain the convergence theorem as follows.

Theorem 5. *Assume the conditions of Theorem* 1 *hold. As the steps* h *and* τ *tend to zero, the solution* $v_\Delta = v_h^\tau = \{v_j^n | j = 0,1,\ldots,J;\ n = 0,1,\ldots,N\}$ *of the difference schemes* (2.4)–(2.6) *converges to the unique generalized solution* $u(x,t) \in W_2^{(2,1)}(Q_T)$ *of the problems* (2.1)–(2.3).

7 Numerical experiment

Consider the following problems

$$
\begin{aligned}
&u_t = uu_{xx} + f(x,t,u,u_x), \quad x \in [0,1], \quad t \in [0,T],\\
&u(0,t) = u(1,t) = 0, \quad t \in (0,T],\\
&u(x,0) = \sin(\pi x), \quad x \in [0,1],
\end{aligned}
$$

where $f(x,t,u,u_x) = \pi^2(e^{-\pi^2 t}\sin(\pi x))^2 + \pi e^{-\pi^2 t}\cos(\pi x) - \pi^2 u - u_x$. The exact solution of the problems is $u = e^{-\pi^2 t}\sin(\pi x)$. We demonstrate the accuracy of the parallel scheme in Table 1.

Table 1 The accuracy of the scheme

$J-1$	20	40	60	80	100
$\max_{j,n}\lvert v_j^n - u_j^n\rvert$	1.7468E-004	4.3880E-005	1.9516E-005	1.0997E-005	7.0508E-006
$\max_{j,n}\frac{\lvert v_j^n - u_j^n\rvert}{\lvert u_j^n\rvert}$	6.4136E-004	2.8055E-004	1.7093E-004	1.1834E-004	8.8271E-005
R_1	–	2	3	4	5
R_2	–	4	9	16	25
R	–	3.98	8.95	15.88	24.77
$J-1$	120	140	160	180	200
$\max_{j,n}\lvert v_j^n - u_j^n\rvert$	4.9082E-006	3.6155E-006	2.7769E-006	2.2018E-006	1.7905E-006
$\max_{j,n}\frac{\lvert v_j^n - u_j^n\rvert}{\lvert u_j^n\rvert}$	6.9135E-005	5.6055E-005	4.6640E-005	3.9593E-005	3.4154E-005
R_1	6	7	8	9	10
R_2	36	49	64	81	100
R	35.59	48.31	62.90	79.34	97.56

When $\tau = 0.0000001$, we compute 100000 times steps, and use four processors. Let R_1 and R_2 be the theoretical values, and R the computational value, whose definition are given as follows. Denote $E_\infty^{J-1} = \max_{\substack{0\leqslant j\leqslant J-1\\ 0\leqslant n\leqslant N}} |v_j^n - u_j^n|$, $h_{J-1} = \frac{1}{J-1}$. If the scheme is of first order accuracy, i.e. $E_\infty^{J-1} \approx C_1(\tau + h_{J-1})$, where C_1 is a constant independent of τ and h_{J-1}, denote $R_1 = \frac{E_\infty^{20}}{E_\infty^{J-1}} \approx \frac{\tau+h_{20}}{\tau+h_{J-1}} \approx \frac{h_{20}}{h_{J-1}} = \frac{J-1}{20}$. If the scheme is of second order accuracy, i.e. $E_\infty^{J-1} \approx C_2(\tau + h_{J-1}^2)$, where C_2 is a constant independent of τ and h_{J-1}, denote $R_2 = \frac{E_\infty^{20}}{E_\infty^{J-1}} \approx \frac{\tau+h_{20}^2}{\tau+h_{J-1}^2} \approx \frac{h_{20}^2}{h_{J-1}^2} = (\frac{J-1}{20})^2$. Denote $R = \frac{E_\infty^{20}}{E_\infty^{J-1}}$. From Table 1, we know that the parallel scheme has the second order accuracy.

In order to demonstrate the unconditional stability of our scheme, we present the numerical results for $r = 10, 100, 1000, 10000$ respectively. We take $J-1 = 100000$, where $r = \tau/h^2$, CPUs is the number of the processor, T_{all} is the total time computing 100000 times steps, S_p is the relative speedup, E_{ff} is the parallel efficiency. From Table 2 to Table 5, we know that our scheme is unconditionally stable, and it has super–linear speedup.

Table 2 The stability and parallelism ($\lambda = 10$)

CPUs	1	10	20	40	80
$\max_{j,n}\lvert v_j^n - u_j^n\rvert$	1.1244E-010	1.1256E-010	1.1254E-010	1.1256E-010	1.1262E-010
$T_{\text{all}}(s)$	21802.29	2051.44	1025.84	517.49	262.47
S_p	1	10.63	21.25	42.13	83.07
$E_{ff}(100\%)$	1	1.06	1.06	1.05	1.04

Table 3 The stability and parallelism ($\lambda = 100$)

CPUs	1	10	20	40	80
$\max_{j,n} \lvert v_j^n - u_j^n \rvert$	1.4152E-009	1.3942E-009	1.3782E-009	1.3404E-009	1.2640E-009
$T_{\mathrm{all}}(s)$	21927.49	2032.37	1031.09	515.35	263.50
S_p	1	10.79	21.27	42.55	83.22
E_{ff}(100%)	1	1.08	1.06	1.06	1.04

Table 4 The stability and parallelism($\lambda = 1000$)

CPUs	1	10	20	40	80
$\max_{j,n} \lvert v_j^n - u_j^n \rvert$	3.8888E-008	2.4748E-008	1.2339E-008	2.0560E-008	7.9452E-008
$T_{\mathrm{all}}(s)$	22923.82	2041.89	1025.28	511.41	262.43
S_p	1	11.23	22.36	44.82	87.35
E_{ff}(100%)	1	1.12	1.12	1.12	1.09

Table 5 The stability and parallelism($\lambda = 10000$)

CPUs	1	10	20	40	80
$\max_{j,n} \lvert v_j^n - u_j^n \rvert$	1.2121E-006	2.0323E-006	4.9454E-006	1.0927E-005	2.2914E-005
$T_{\mathrm{all}}(s)$	22123.62	2076.82	1024.24	514.36	267.47
S_p	1	10.65	21.60	43.01	82.71
E_{ff}(100%)	1	1.07	1.08	1.08	1.03

8 Some specific parallel schemes with interface extrapolation and future plans

8.1 We can get some specific parallel schemes by giving the parameter in the general scheme (2.4). And we will solve the parabolic initial-boundary value problem arising from a large scale scientific and engineering computation by using these schemes on parallel computers.

Take the following four kinds of differences for the second-order derivative v_{xx}: (i) Central (implicit) scheme $\mathcal{C}$

$$\frac{v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1}}{h^2},$$

(ii) right scheme $\mathcal{R}$

$$\frac{(2v_{j+1}^n - v_{j+1}^{n-1}) - 2v_j^{n+1} + v_{j-1}^{n+1}}{h^2},$$

(iii) left scheme $\mathcal{L}$

$$\frac{v_{j+1}^{n+1} - 2v_j^{n+1} + (2v_{j-1}^n - v_{j-1}^{n-1})}{h^2},$$

(iv) explicit scheme with Jacobian type $\mathcal{J}$

$$\frac{(2v_{j+1}^n - v_{j+1}^{n-1}) - 2v_j^{n+1} + (2v_{j-1}^n - v_{j-1}^{n-1})}{h^2}.$$

In general, there are $0 < \lambda_j \leqslant 1$ and $0 < \mu_j \leqslant 1$ for the inner point $\{x_j\}$ in sub-domains. Usually take $\lambda_j = 1$ and $\mu_j = 1$, i.e. the scheme $\mathcal{C}$. Then, the scheme in the inner of sub-domains can be expressed as

$$\mathcal{LCC}\cdots\mathcal{CCR}.$$

Let j_0 and n be two fixed integers such that $1 < j_0 < J-1$, $1 \leqslant n \leqslant N$. Let $x = x_{j_0}$ be the inner boundary point between the sub-domains.

Example 1. If $\lambda_{j_0} = 0$, then the computation of $\{v_j^{n+1} : j \leqslant j_0\}$ in the $(n+1)$-layer is independent of $\{v_j^{n+1} : j \geqslant j_0+1\}$. The scheme near to x_{j_0} can be expressed as

$$\cdots \mathcal{CCCRCCC} \cdots$$

Similarly, if $\mu_{j_0+1} = 0$, Then the computation of $\{v_j^{n+1} : j \geqslant j_0+1\}$ in the $(n+1)$-layer is independent of $\{v_j^{n+1} : j \leqslant j_0\}$. The scheme near to x_{j_0} can be expressed as

$$\cdots \mathcal{CCCLCCC} \cdots$$

Example 2. If there are $\lambda_{j_0} = 0$ and $\mu_{j_0+1} = 0$, then $\{v_j^{n+1} : j \leqslant j_0\}$ and $\{v_j^{n+1} : j \geqslant j_0+1\}$ in the $(n+1)$–layer can be simultaneously computed. The scheme near to x_{j_0} can be expressed as

$$\cdots \mathcal{CCRLCC} \cdots .$$

The designing of this scheme is simple, and it can be readily implemented in the parallel computer, but it does not satisfy $\lambda_j + \mu_{j+1} \geqslant 1$.

Example 3. If there are $\lambda_{j_0-1} = 0$, $\mu_{j_0+1} = 0$ and $\lambda_{j_0} = \mu_{j_0} = 1$, first we simultaneously compute $\{v_j^{n+1} : j \leqslant j_0-1\}$ and $\{v_j^{n+1} : j \geqslant j_0+1\}$ in the $(n+1)$-layer, then using the values $v_{j_0-1}^{n+1}$ and $v_{j_0+1}^{n+1}$ obtained in the first step, we explicitly compute $v_{j_0}^{n+1}$ by using the implicit scheme. The scheme near to x_{j_0} can be expressed as

$$\cdots \mathcal{CCRCLCC} \cdots$$

The designing of this scheme is simple, and it can be readily implemented in the parallel computer, moreover it satisfies $\lambda_j + \mu_{j+1} \geqslant 1$.

Example 4. If there are $\lambda_{j_0} = \mu_{j_0} = 0$, and $\lambda_j = \mu_j = 1$ for others, we first compute the value of $v_{j_0}^{n+1}$, then using the value of $v_{j_0}^{n+1}$ obtained, we can simultaneously compute the values of $\{v_j^{n+1} : j \leqslant j_0-1\}$ and $\{u_j^{n+1} : j \geqslant j_0+1\}$. The scheme near to x_{j_0} can be expressed as

$$\cdots \mathcal{CCCJCCC} \cdots .$$

The designing of this scheme is simple, and it can be readily implemented in the parallel computer, moreover it satisfies $\lambda_j + \mu_{j+1} \geqslant 1$.

8.2 we can extend the results in this paper to the multi-dimensional nonlinear parabolic problems and the parabolic systems of the divergence type.

It is most important for practical applications to investigate the conservation of the parallel schemes, which has not been discussed except the sub-implicit scheme in [12]. Generally speaking, the fully implicit scheme is the conservative scheme, but the conservation will be lost when some schemes with the explicit character at the inner interface and the implicit scheme only in the interior of sub-domain are used. Therefore the conservation (the continuity of flux) should be kept at the inner interface. We must modify the values at the inner interface by using the implicit scheme based on the available parallel scheme. For example, for the parallel scheme given in Example 4, first it predicts the values (denoted by $\bar{v}_k^{n+1}$) of the inner interface by taking the explicit scheme with the Jacobian type at the inner interface between the sub-domains, then simultaneously solves the values in the sub-domains by the implicit scheme so that the values of v_{k+1}^{n+1} and v_{k-1}^{n+1} are obtained, and at last the value of $\bar{v}_k^{n+1}$ is modified to give the value of v_k^{n+1} by a conservative scheme as follows:

$$\frac{v_k^{n+1} - v_k^n}{\tau} = \frac{1}{h^2}(v_{k+1}^{n+1} - \bar{v}_k^{n+1} - (\bar{v}_k^{n+1} - v_{k-1}^{n+1})).$$

We will extensively discuss the construction of the conservative parallel schemes in future.

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